Active Filters

ELEN-457

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Spring 2003

Filters

- Introduction
- Low-Gain Filters
- Infinite Gain Filters
- State Variable Filters
- Ladder Filters
- Other techniques (Follow the leader, GICs)

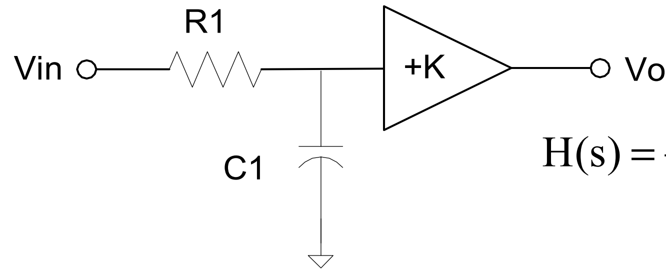
Introduction: First-order Filter

$$H(s) = \frac{K1}{s + \omega_0}$$

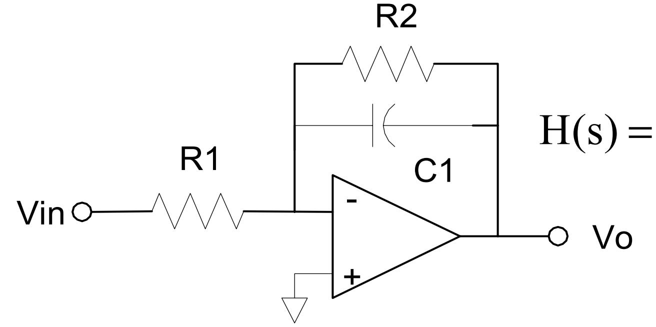
$$H(j\omega) = \frac{K1}{(\omega_0) + j(\omega)}$$

$$|H(j\omega)| = \frac{K1}{\sqrt{(\omega_0)^2 + (\omega)^2}}$$

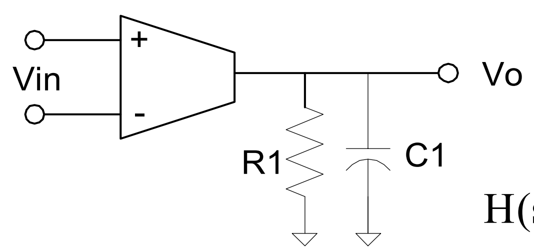
$$\text{Phase}(H(j\omega)) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$



$$H(s) = \frac{K}{1 + sR_1C_1}$$

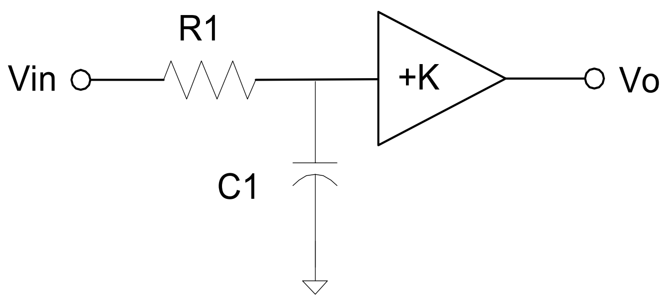


$$H(s) = -\frac{\frac{R_2}{R_1}}{1 + sR_2C_1}$$

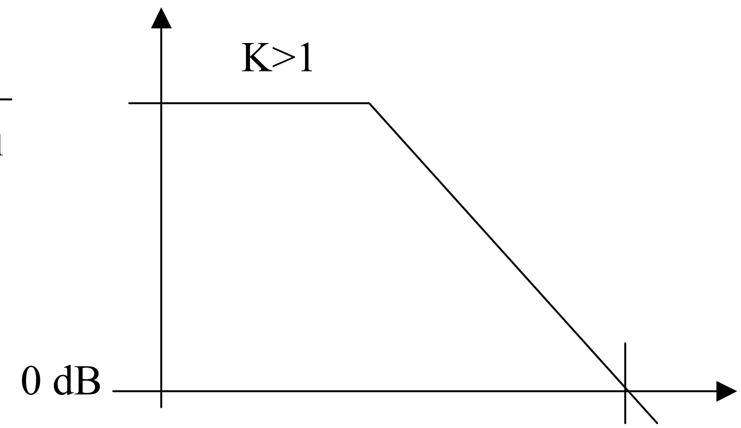


$$H(s) = -\frac{g_m R_1}{1 + sR_1C_1}$$

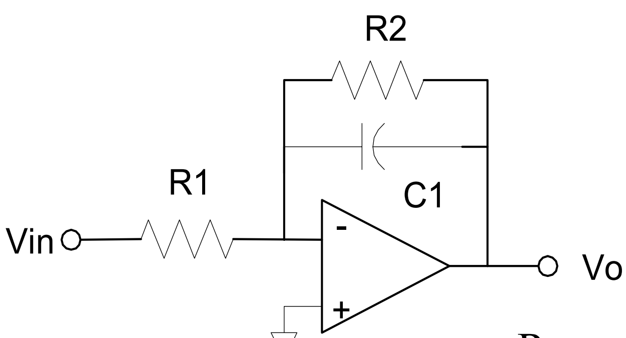
Introduction: First-order Filter



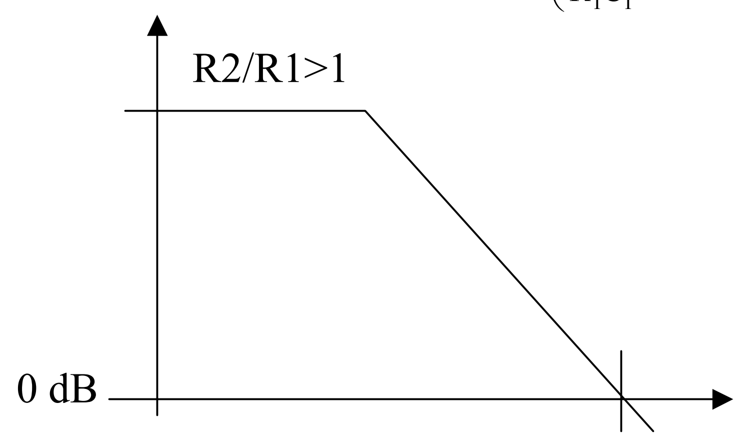
$$H(s) = \frac{K}{1 + sR_1C_1}$$



$$\omega_u = \left(\frac{1}{R_1C_1}\right)\sqrt{K^2 - 1}$$

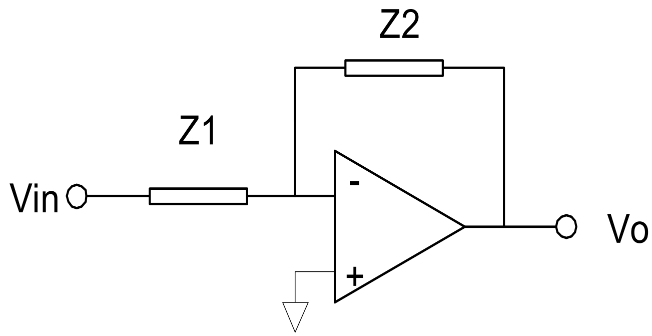


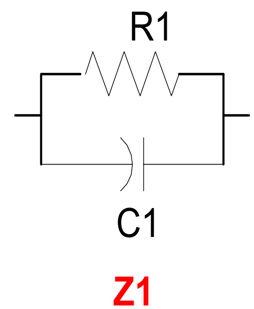
$$H(s) = -\frac{\frac{R_2}{R_1}}{1 + sR_2C_1}$$

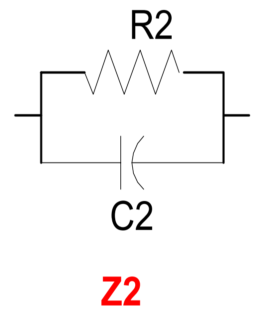


$$\omega_u = \left(\frac{1}{R_2C_1}\right)\sqrt{\left(\frac{R_2}{R_1}\right)^2 - 1}$$

Introduction: First-order Filter



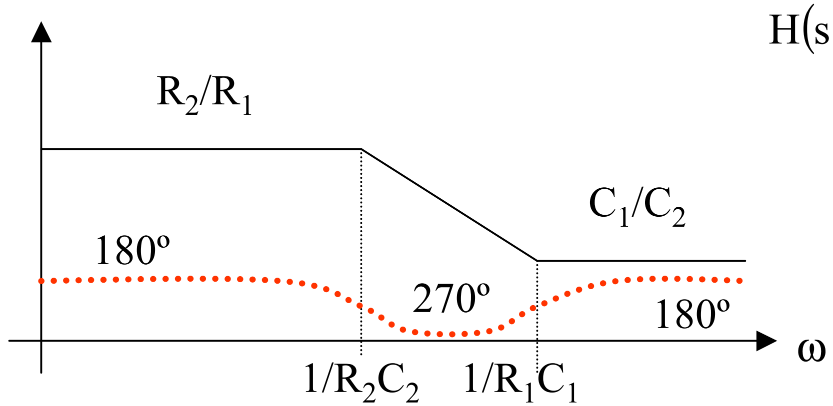




$$H(s) = -\frac{Z2}{Z1} = -\frac{Y1}{Y2}$$

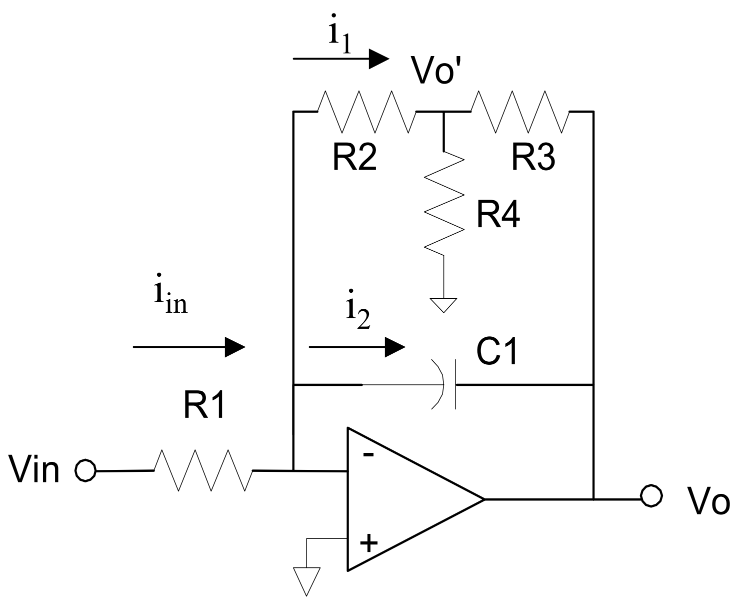
$$Y1 = g_1 + sC_1 \qquad Y2 = g_2 + sC_2$$

$$H(s) = -\frac{g_1 + sC_1}{g_2 + sC_2} = -\left(\frac{R_2}{R_1}\right)\left(\frac{1 + sR_1C_1}{1 + sR_2C_2}\right)$$



- **LH Pole => negative (lag) excess phase**
- **LH Zero => positive (lead) excess phase**

Introduction: Large resistor implementation



Notice that $$Vo' = \frac{R_2 \| R_4}{R_3 + (R_2 \| R_4)} V_0 = \alpha V_0$$

$$i_{in} = g_1 v_{in}$$

$$i_1 = -g_2 v_0' \quad ; \quad i_2 = -sC_1 v_0$$

$$i_{in} = g_1 v_{in} = (\alpha g_2 + sC_1) v_0$$

$$H(s) = -\frac{g_1}{\alpha g_2} \frac{1}{1 + s\left(\frac{C_1}{\alpha g_2}\right)} = -\frac{R_2}{\alpha R_1} \frac{1}{1 + s\left(\frac{R_2 C_1}{\alpha}\right)}$$

$$\alpha = \frac{R_2 R_4}{R_3(R_2 + R_4) + (R_2 R_4)} = \frac{1}{\left(\frac{R_3}{R_2} + \frac{R_3}{R_4}\right) + 1}$$

- **Effective R_2 is R_2/α**
- **The smaller α the larger the equivalent resistance**
- **The attenuation factor α can be easily reduced**

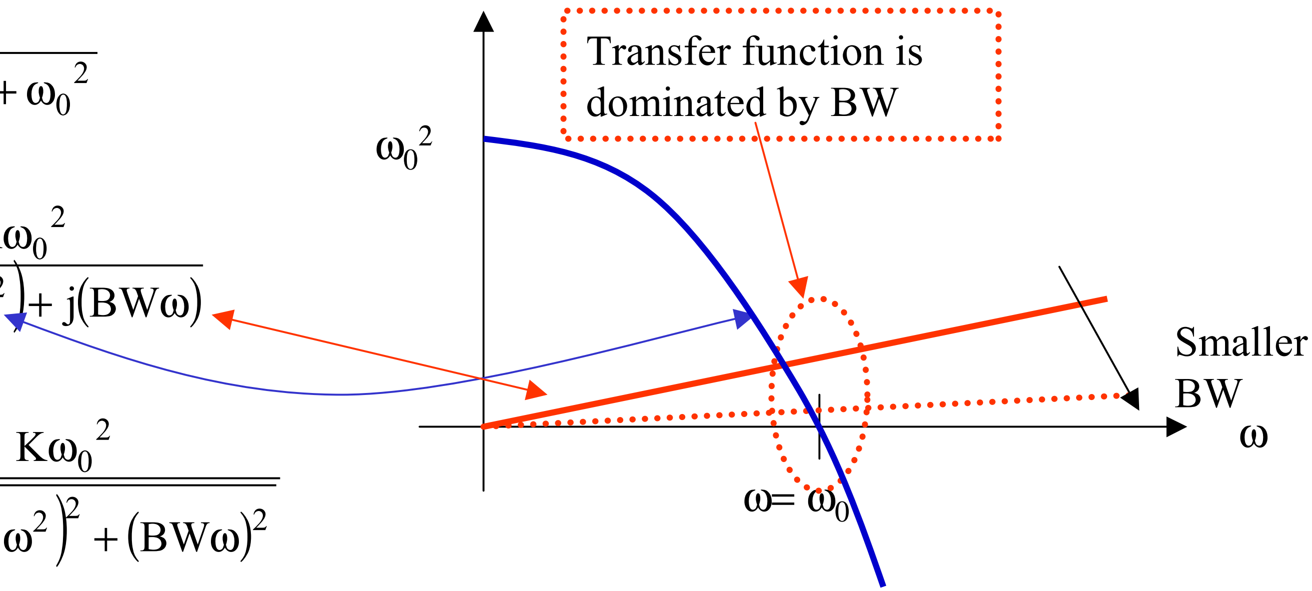
Biquadratic Function: Frequency Domain Analysis

$$H(s) = \frac{K\omega_0^2}{s^2 + BWs + \omega_0^2}$$

$$H(j\omega) = \frac{K\omega_0^2}{\left(\omega_0^2 - \omega^2\right) + j(BW\omega)}$$

$$\left|H(j\omega)\right| = \frac{K\omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (BW\omega)^2}}$$

$$\text{Phase}(H(j\omega)) = \tan^{-1}\left(\frac{\text{im}(H(j\omega))}{\text{real}(H(j\omega))}\right)$$



Any small parameter variation around $\omega=\omega_0$ produces huge variations on the overall transfer function.

Narrow-band filters (small BW) are more critical!!

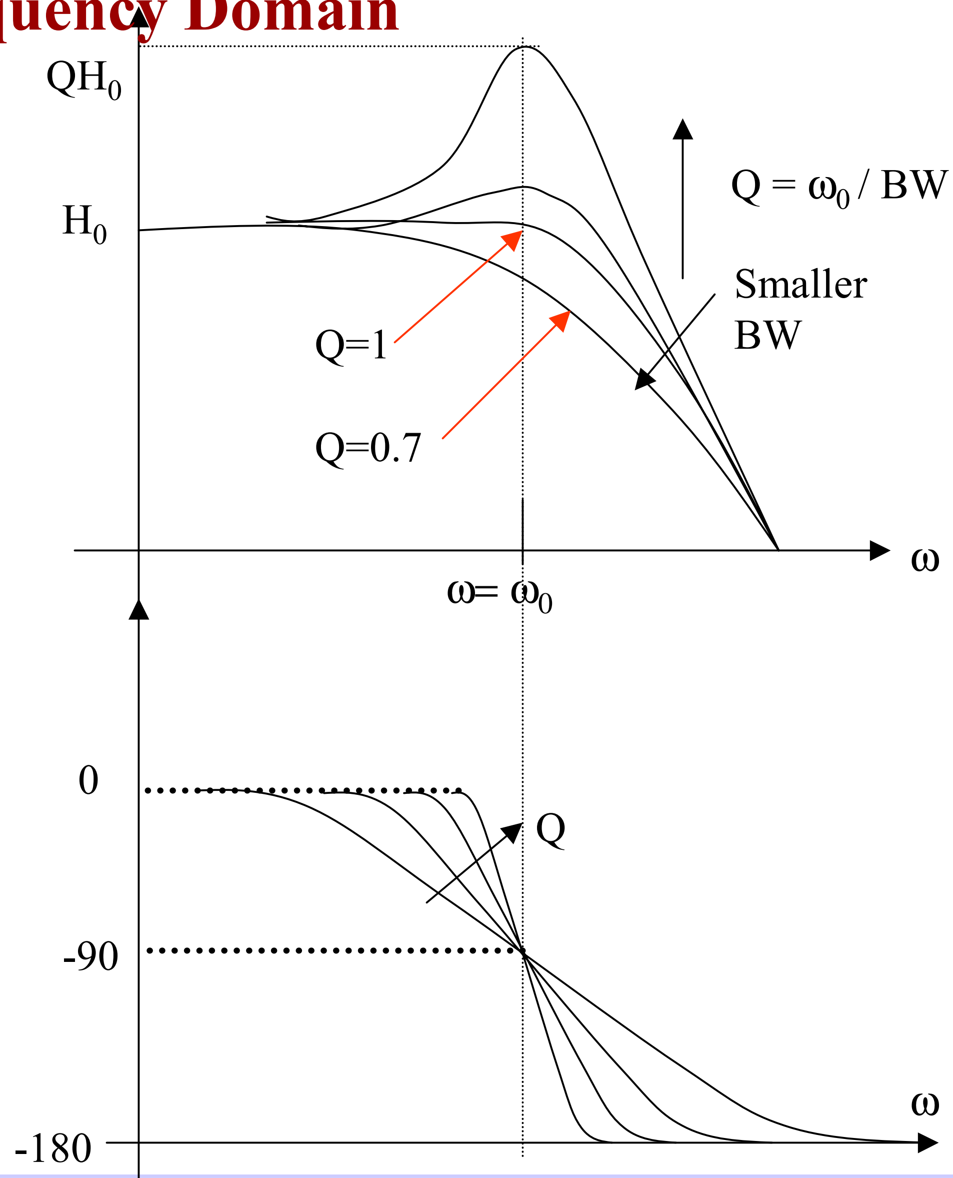
Frequency Domain

$$H(s) = \frac{K\omega_0^2}{s^2 + BWs + \omega_0^2}$$

$$|H(j\omega)| = \frac{K\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (BW\omega)^2}}$$

$$\text{Phase}(H(j\omega)) = \tan^{-1}\left(\frac{\text{im}(H(j\omega))}{\text{real}(H(j\omega))}\right)$$

$$\text{Phase}(H(j\omega)) = -\tan^{-1}\left(\frac{BW\omega}{\omega_0^2 - \omega^2}\right)$$



SENSITIVITY

$$^{f}_{x} = \left(\frac{x}{f}\right)\frac{\partial}{\partial x} f \cong \left(\frac{x}{\Delta x}\right)\left(\frac{\Delta f}{f}\right) = \frac{\frac{\Delta f}{f}}{\frac{\Delta x}{x}}$$

Practical implication =>

$$\boxed{\frac{\Delta f}{f} = \left(^{f}_{x}\right)\frac{\Delta x}{x}}$$

If you know the sensitivity function, then you can evaluate the effect on f of variations in parameter x!

Notice that sensitivity relates the normalized (percentage) errors; e.g. for a sensitivity function of 10, variations of 10% in x produces f variations of 100%. Sensitivity of 0.1 will produce a variation of 1%.

SENSITIVITY: properties

$${}^{kf}_{x} = {}^{f}_{kx} = {}^{f}_{x}$$

$${}^{kx}_{x} = {}^{x}_{kx} = 1$$

$${}^{f^n}_{x} = n\,{}^{f}_{x}$$

$${}^{x^n}_{x} = n$$

$${}^{f}_{x^n} = \left(\frac{1}{n}\right){}^{f}_{x}$$

$${}^{\frac{f}{g}}_{x} = {}^{f}_{x} - {}^{g}_{x}$$

$${}^{\Pi(f_1..f_n)}_{x} = \sum\left({}^{f_1}_{x} + .. + {}^{f_n}_{x}\right)$$

$${}^{f_i}_{x} = \frac{\left(f_i\,{}^{f_i}_{x}\right)}{f_i}$$

EXAMPLE1:

$$BW = \frac{1}{R_1C_1} + \frac{1}{R_2C_2}$$

$${}^{BW}_{R_1} = \frac{\frac{1}{R_1C_1}\,{}^{\frac{1}{R_1C_1}}_{R_1}}{BW} = -\frac{\frac{1}{R_1C_1}}{BW} = -\frac{1}{1+\frac{R_1C_1}{R_2C_2}}$$

EXAMPLE2: Sallen&Key

$$BW = \frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_2C_2}(1-K)$$

$${}^{BW}_{K} = \frac{\frac{1-K}{R_2C_2}\,{}^{1-K}_{K}}{BW} = \frac{\frac{1-K}{R_2C_2}}{BW}\left(\frac{-K}{1-K}\right)$$

$$= -\frac{K}{\frac{R_2C_2}{R_1C_1} + \frac{C_2}{C_1} + (1-K)}$$

SENSITIVITY: properties

EXAMPLE3:

$$BW = \frac{g_{eq}}{C_1} = \frac{g_1 + g_2}{C_1} \qquad {}^{BW}_{g_1} = \frac{g_1}{g_1 + g_2} < 1$$

Little sensitive

$$BW = \frac{g_{eq}}{C_1} = \frac{g_1 - g_2}{C_1} \qquad {}^{BW}_{g_1} = \frac{g_1}{g_1 - g_2} = \frac{1}{1 - \frac{g_2}{g_1}} > 1$$

Very sensitive to conductance mismatches
This is the case if you use negative resistors for gain enhancement!!

The use of negative elements is attractive because you can drastically reduce the spread of the components (due to the subtraction). Unfortunately the sensitivity increases drastically!!

➨ **TRADEOFF: Spread ⟺ Sensitivity**

For low sensitivity

$$\boxed{{}^{f}_{x} \leq 1}$$

SENSITIVITY: properties

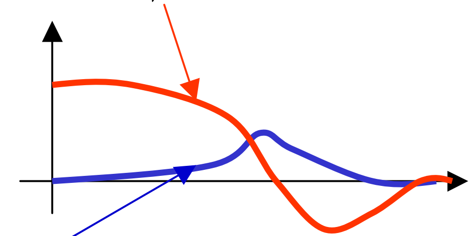
EXAMPLE4:

$$H(s) = \frac{K\omega_0^{\,2}}{s^2 + BWs + \omega_0^{\,2}}$$

$$\left|H(j\omega)\right| = \frac{K\omega_0^{\,2}}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + (BW\omega)^2}}$$

$$S_{\omega_0}^{\left|H(j\omega)\right|} = 2 - \frac{1}{2}\left(\frac{\left(\omega_0^{\,2} - \omega^2\right)^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (BW\omega)^2}\right)\left(\frac{4\omega_0^{\,2}}{\omega_0^{\,2} - \omega^2}\right) = 2\left(1 - \frac{\left(\omega_0^{\,2} - \omega^2\right)\omega_0^{\,2}}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (BW\omega)^2}\right)$$

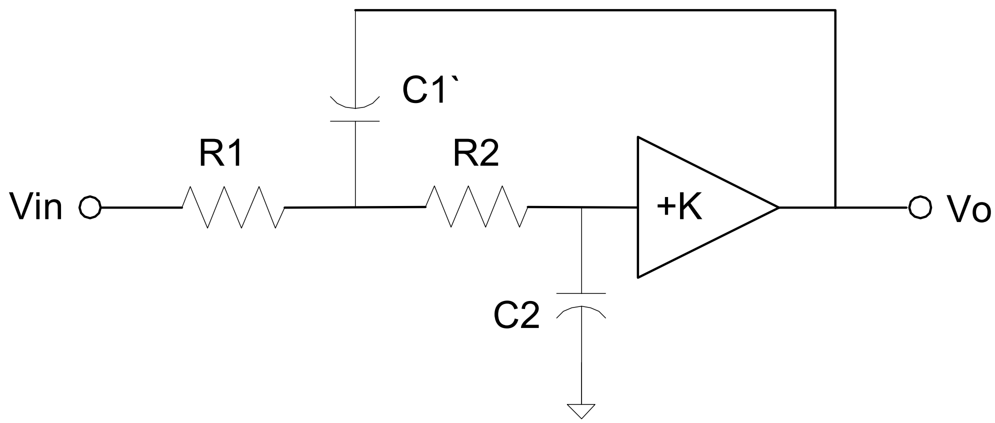
$$= 2\left(\frac{(BW\omega)^2 - \left(\omega_0^{\,2} - \omega^2\right)\omega^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (BW\omega)^2}\right)$$



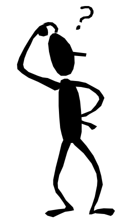
$$S_{BW}^{\left|H(j\omega)\right|} = \frac{1}{2}\left(\frac{2(BW\omega)^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (BW\omega)^2}\right) = \frac{(BW\omega)^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (BW\omega)^2}$$

These results are general!!

Low-Gain Filters: Sallen-Key



$$H(s)=\frac{K\left(\frac{1}{R_1R_2C_1C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)\right)s+\frac{1}{R_1R_2C_1C_2}}$$



- Why this topology
- behaves as LPF?

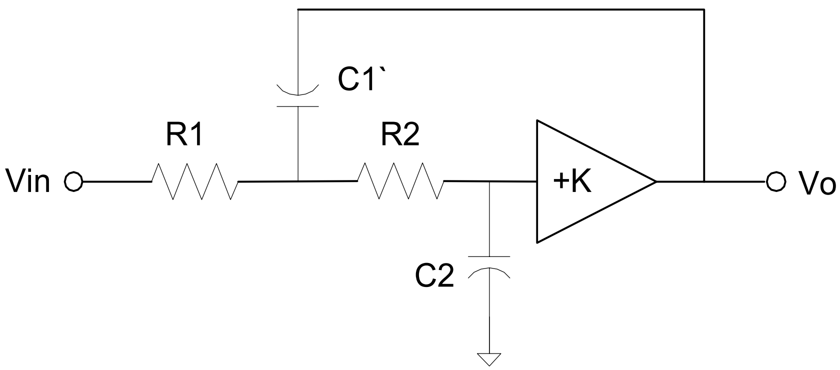
- How stable is this circuit if K is precise?

- Practical Limits?

- How to design a filter?

At ω=0, capacitors are open circuits and H(s)=K
At $\omega\sim\omega_0$, all impedances are important
At $\omega=\infty$, capacitors are dominant

Low-Gain Filters: Sallen-Key



$$H(s)=\frac{K\left(\frac{1}{R_1R_2C_1C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)\right)s+\frac{1}{R_1R_2C_1C_2}}$$

Lowpass

$$H(0)=K$$

$$\omega_0{}^2=\frac{1}{R_1R_2C_1C_2}$$

$$H(\omega=\omega_0)=\frac{K}{\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)\right)}$$

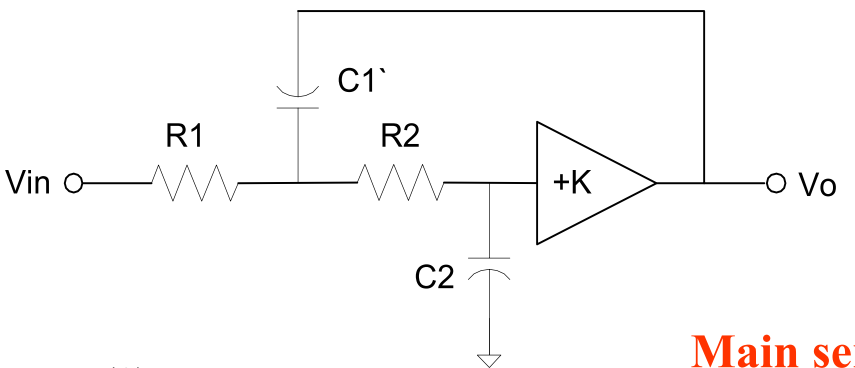
$$BW=\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)$$

At ω=0, H(s)=K

At $\omega\sim\omega_0$, all impedances are important

Meaning of -K?   Positive feedback!!!

Low-Gain Filters: Sallen-Key



$$H(s)=\frac{K\left(\frac{1}{R_1R_2C_1C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)\right)s+\frac{1}{R_1R_2C_1C_2}}$$

$$H(0)=K$$

$$\omega_0{}^2=\frac{1}{R_1R_2C_1C_2}$$

$$H(\omega=\omega_0)=\frac{K\omega_0}{\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)\right)}$$

$$BW=\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}(1-K)$$

Main sensitivities:

$$S_K^{|H(j0)|}=1$$

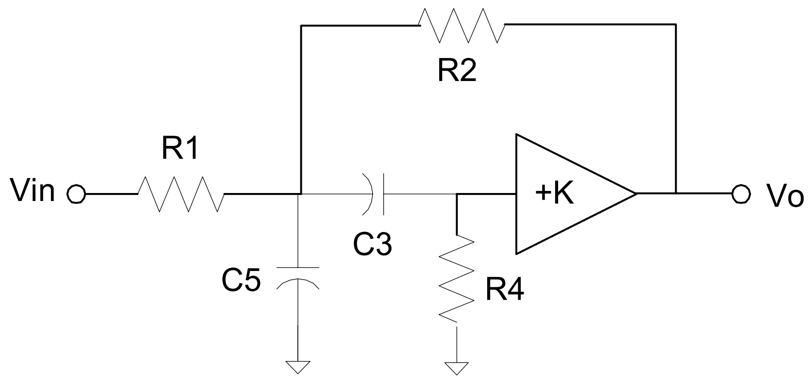
$$S_{R_{1,2},C_{1,2}}^{\omega_0}=-\frac{1}{2}$$

$$S_K^{|H(\omega_0)|}=1-\frac{1}{BW}\left(\frac{-K}{R_2C_2}\right)$$

$$=1+\frac{1}{\frac{R_2C_2}{R_1C_1}+\frac{C_2}{C_1}+1-K}$$

Can you design low sensitive narrow-band filters?
If not, why not?

Low-Gain Filters: Sallen-Key BP filter



$$\omega_0^{\,2} = \frac{1}{R_4 C_3 C_5}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$H(\omega = \omega_0) = \frac{K\left(\frac{1}{R_1 C_5}\right)}{\frac{1}{R_1 C_5} + \frac{1}{R_4 C_5} + \frac{1}{R_4 C_3} + \frac{1-K}{R_2 C_5}}$$

$$BW = \frac{1}{R_1 C_5} + \frac{1}{R_4 C_5} + \frac{1}{R_4 C_3} + \frac{1-K}{R_2 C_5}$$

$$H(s) = \frac{sK\left(\frac{1}{R_1 C_5}\right)}{s^2 + \left(\frac{1}{R_1 C_5} + \frac{1}{R_4 C_5} + \frac{1}{R_4 C_3} + \frac{1-K}{R_2 C_5}\right)s + \frac{1}{R_4 C_3 C_5}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

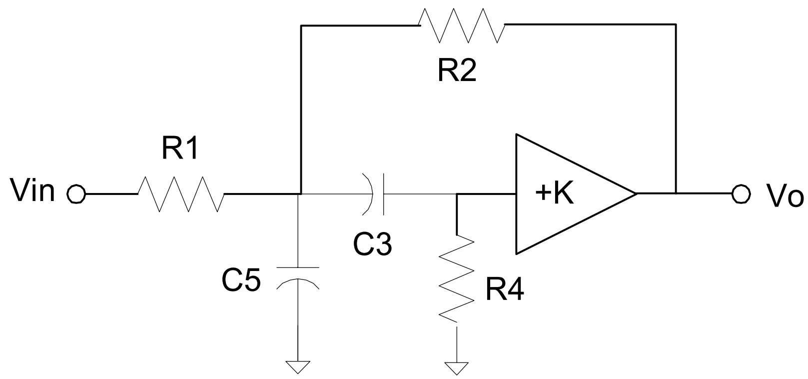
$${}^{|H(j0)|}_{K} = 1$$

$${}^{\omega_0}_{R_4, C_{3,5}} = -\frac{1}{2}; \qquad {}^{\omega_0}_{R_1, R_2} < -\frac{1}{2}$$

$${}^{BW}_{K} = \frac{1}{BW}\left(\frac{-K}{R_2 C_5}\right) = \frac{1}{\frac{R_2}{R_1} + \frac{R_2}{R_4} + \frac{R_2 C_5}{R_4 C_3} + 1 - K}$$

Typical design approach : C3=C5, K=2

Low-Gain Filters: Sallen-Key BP filter



$f_0 = 10\text{kHz}$

$BW = 1\text{kHz}$

$Q = 10$

Peak gain?

Typical design approach : C3=C5, R1=R2=R3

$$\omega_0^2 = \frac{2}{R_1^2 C_3^2}$$

$$H_{peak} = \frac{K}{4-K}$$

$$Q = \frac{\sqrt{2}}{4-K}$$

$$S_K^{BW} = \frac{1}{4-K}$$

$R_1C_3 = 2.25\text{x}10^{-5}$

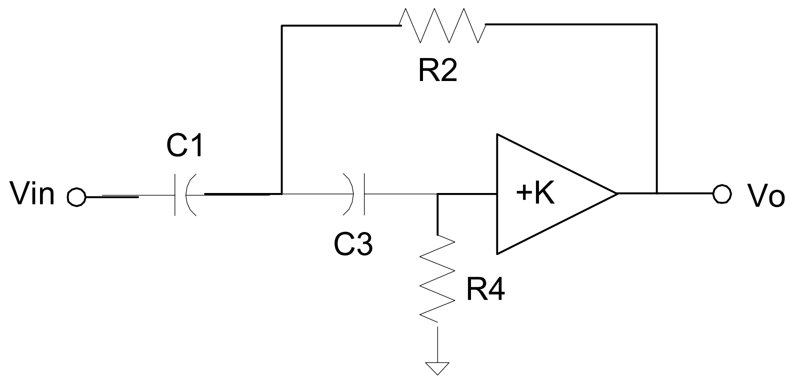
K=3.8565

$H_{peak} \sim 27$!!

$S_K^{BW} = 7.07$!!

Can you design same filter using K=2? Sensitivities?

Low-Gain Filters: Sallen-Key HP filter



$$H(s) = \frac{s^2 K}{s^2 + \left(\frac{1}{R_4 C_3} + \frac{1}{R_4 C_1} + \frac{1-K}{R_2 C_1} \right) s + \frac{1}{R_2 R_4 C_1 C_3}}$$

$$\omega_0^{\ 2} = \frac{1}{R_2 R_4 C_1 C_3}$$

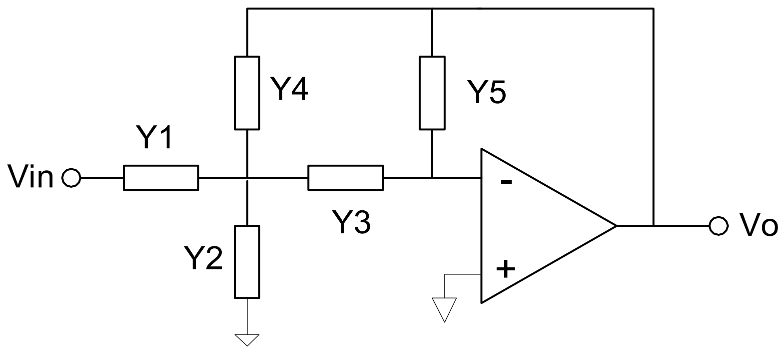
$$H(\omega = \infty) = K$$

$$H(\omega = \omega_0) = \frac{K\omega_0}{\frac{1}{R_4 C_3} + \frac{1}{R_4 C_1} + \frac{1-K}{R_2 C_1}}$$

$$BW = \frac{1}{R_4 C_3} + \frac{1}{R_4 C_1} + \frac{1-K}{R_2 C_1}$$

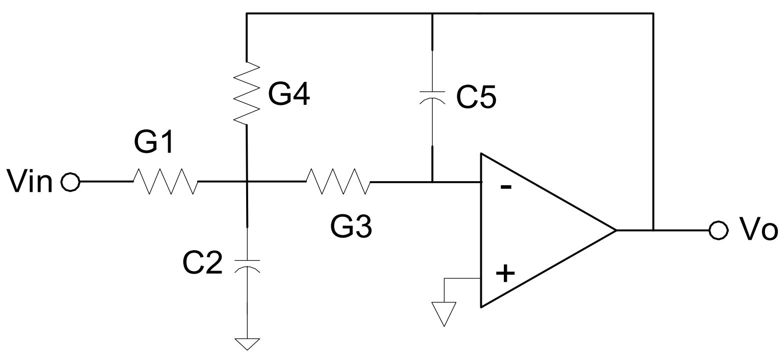
Sensitivities are similar for all these topologies

High-Gain Filters: Multiple Feedback



$$H(s) = \frac{-Y_1 Y_3}{Y_5 \left(Y_1 + Y_2 + Y_3 + Y_4\right) + Y_3 Y_4}$$

- **For LP filter: Y_1 and Y_3 must be conductance and Y_5 must be capacitor**
- **For BP filter: Y_1 or Y_3 must be capacitors**
- **For HP filter: Y_1, Y_3 and either Y4 or Y5 must be capacitors as**
- **Notice that all sensitivities are less than 1! (why?)**



Low-pass filter

$$H(s) = \frac{-G_1 G_3}{s^2 C_5 C_2 + s C_5 \left(G_1 + G_3 + G_4\right) + G_3 G_4}$$

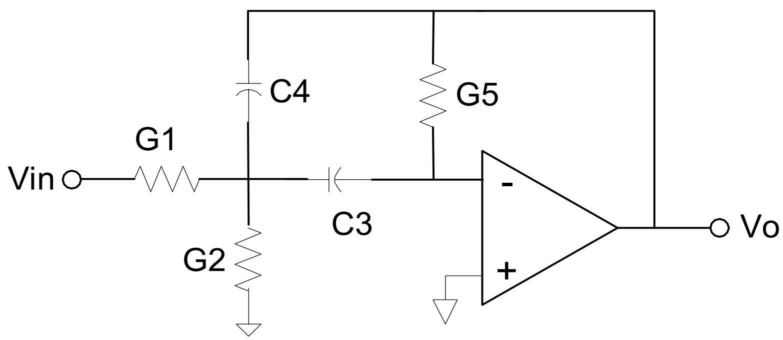
$$\omega_0^2 = \frac{1}{C_5 C_2 R_3 R_4}$$

$$BW = \frac{1}{\left(R_1 \| R_3 \| R_4\right) C_2}$$

All sensitivities are = < 1 !!!

High-Gain Filters: Multiple Feedback

$$H(s) = \frac{-Y_1 Y_3}{Y_5\left(Y_1 + Y_2 + Y_3 + Y_4\right) + Y_3 Y_4}$$



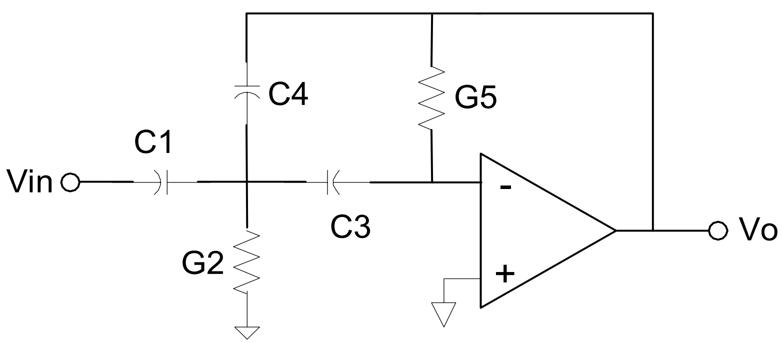
Band-pass filter

$$H(s) = \frac{-sG_1C_3}{s^2C_3C_4 + sG_5\left(C_3 + C_4\right) + G_2G_5}$$

$$\omega_0{}^2 = \frac{1}{C_5C_2R_2R_5}$$

$$BW = \frac{1}{\left(\frac{C_3C_4}{C_3 + C_4}\right)R_5}$$

Sensitivities are =< 1



High-pass filter

$$H(s) = \frac{-s^2C_1C_3}{s^2C_3C_4 + sG_5\left(C_1 + C_3 + C_4\right) + G_2G_5}$$

$$\omega_0{}^2 = \frac{1}{C_3C_4R_2R_5}$$

$$BW = \frac{1}{\left(\frac{C_3C_4}{C_1 + C_3 + C_4}\right)R_5}$$

Sensitivities are =< 1