ECE 458

Analog and Mixed-Signal Center

Filter Approximations & Frequency Transformations

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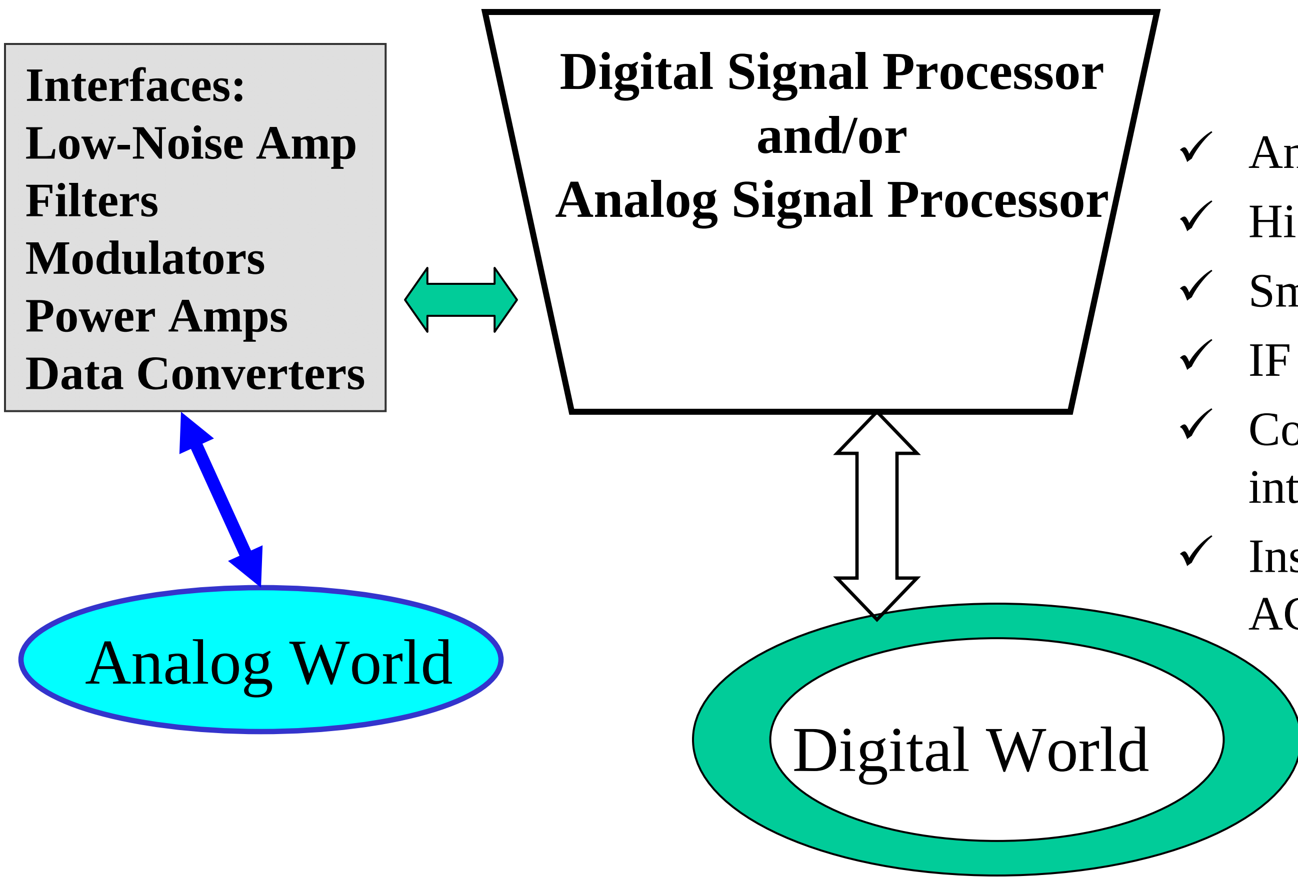
WHY ANALOG FILTERS?



Analog Filter Applications

- ✓ Antialias filters
- ✓ High-Performance Filtering
- ✓ Smoothing
- ✓ IF and Image Rejection
- ✓ Converters (are based on integrators)
- ✓ Instrumentation (PLL´s, AGC´s, control loops, etc.)

Filter Design Conventional Procedures

- **Transform your filter specs into a normalized LPF**

- **Filter order, zeros, poles and/or values for the passive elements can be obtained from tables or from a software package like FIESTA**

- **If you use biquadratic sections, you need poles and zeros matching**

- **For ladder filters, the networks can be obtained from tables**

- **Transform the normalized transfer function to your filter by using**
  - **Filter transformation (LP to BP, HP, BR)**
  - **Frequency transformation**
  - **Impedance denormalization**

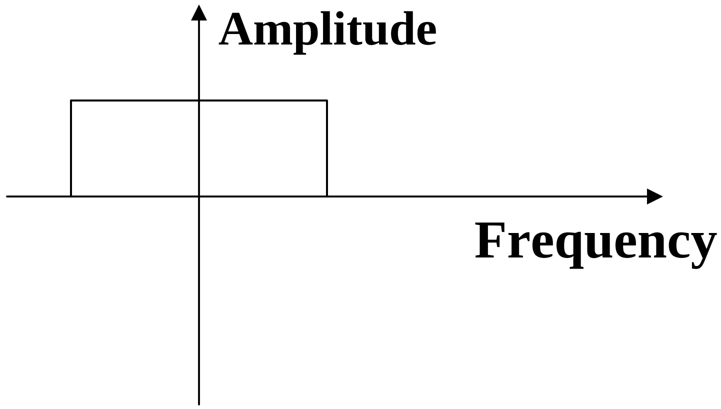
- **You obtain the transfer function or your passive network**

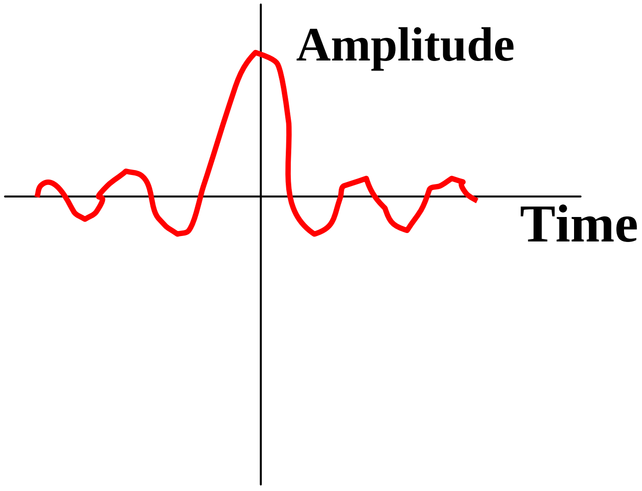
Filter Approximation Concepts

How do you translate filter specifications into a mathematical expression which can be synthesized ?

- Approximation Techniques

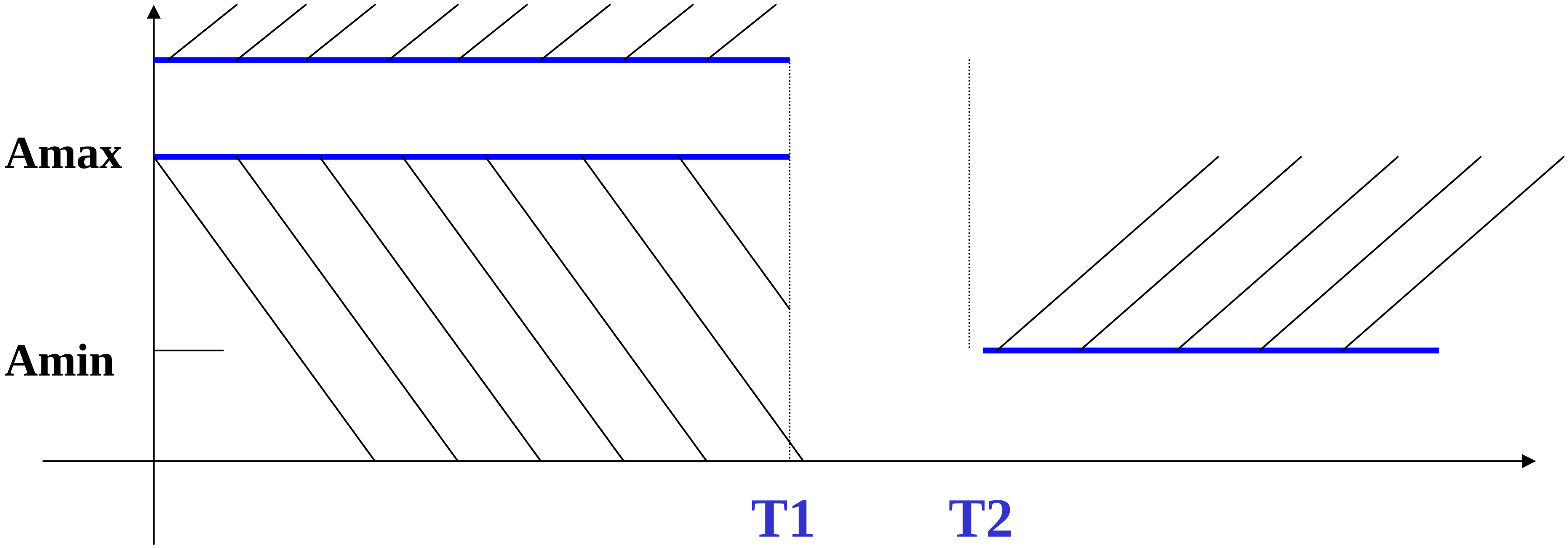
Why an ideal Brick Wall Filter can not be implemented ?





Filter Approximation Concepts

Practical Implementations are given via window specs.



Amin = Ap is the minimum attenuation in the stopband

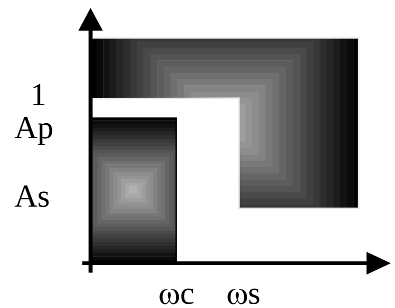
Amax = As is the maximum attenuation in the passband

T1-T2 is the Transition Width

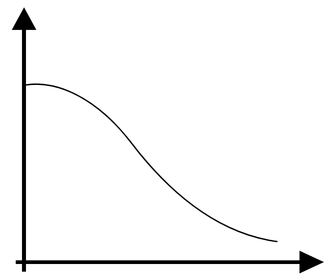
APPROXIMATION TYPES OF LOWPASS FILTERS

Definitions

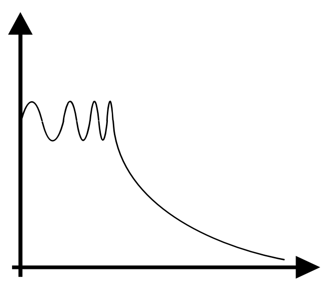
- Ripple = 1-Ap
- Stopband attenuation =As
- Passband (cutoff frequency) = ωc
- Stopband frequency = ωs



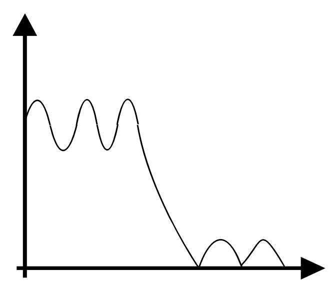
Filter specs



Maximally flat

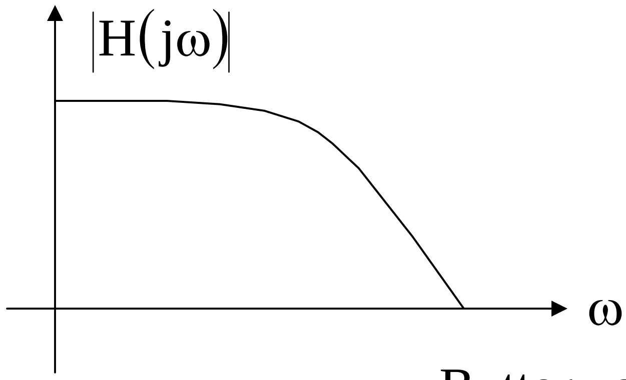


Equal-ripple

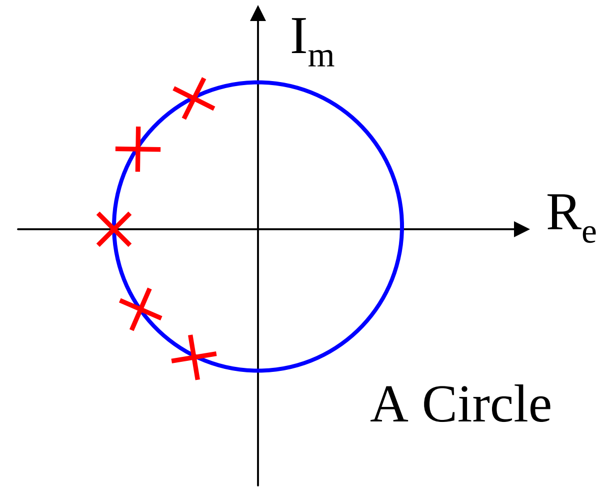


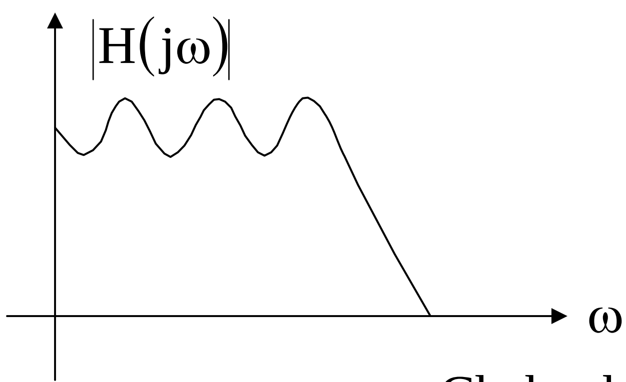
Elliptic

Conventional Filter Approximations

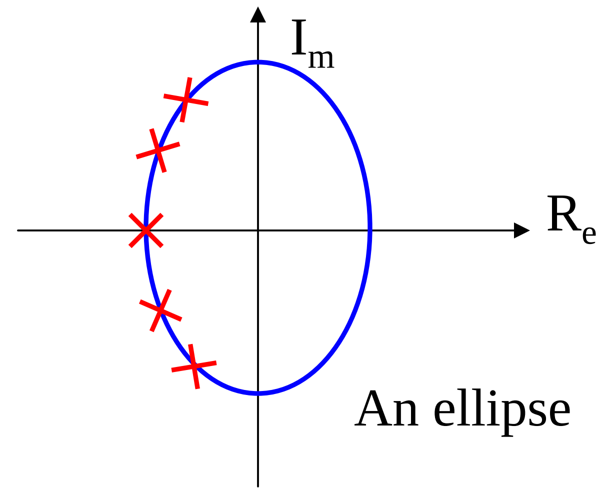


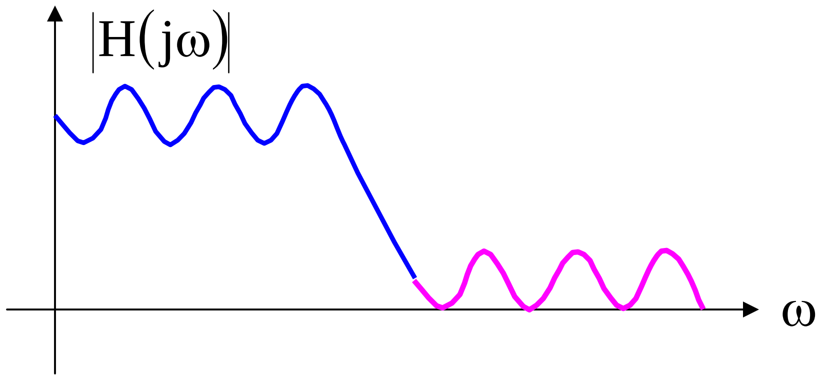
Butterworth
(Maximum Flat)

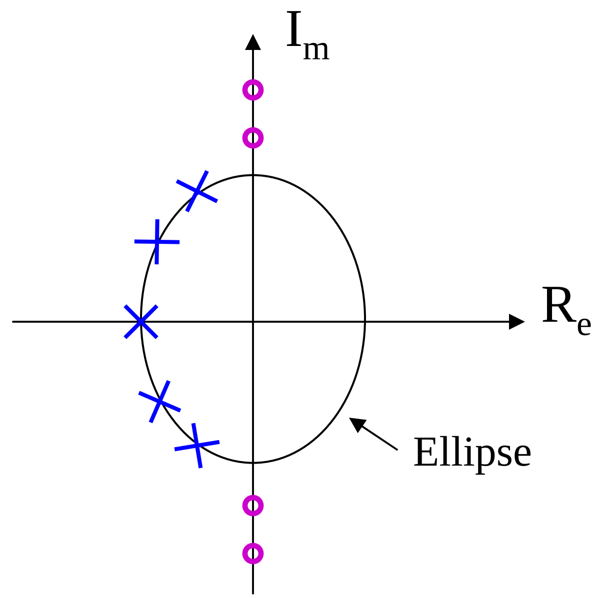




Chebychev
(Equal Ripple Pass Band)







Elliptic (Equal ripple passband and stopband)

Remarks

- Filter approximation meeting the same specifications yield
  Order (Butterworth) > Order (Chebyshev) > Order (Elliptic)

- Phase and group delay also play important roles in many applications

Summary

- Magnitude Approximations:
  - -- Butterworth
  - -- Chebyshev (Direct or Inverse)
  - -- Elliptic

- The group delay of the different magnitude approximations meeting the same (magnitude) specifications vary significantly. The most (least) non-linear phase is associated with the Elliptic (Butterworth) approximation

- Linear Phase and Constant Group Delay Approximation:

  Bessel (Thompson)

- Many Filter Approximations well documented in software programs are available, among them:

  FILSYN, Analog-Artist (Cadence), Math Lab

  Fiesta-II (At TAMU)

Properties of Stable Network Functions

$$N(s)=\frac{A(s)}{B(s)}=K\frac{1+b_1s+b_2s^2+..}{1+a_1s+a_2s^2+..}$$

- Typically the transfer function presents the form of a ratio of two polynomials
- For **non-negative** elements the **coefficients are real and positive**
- The poles are located in the left side of the s-plane

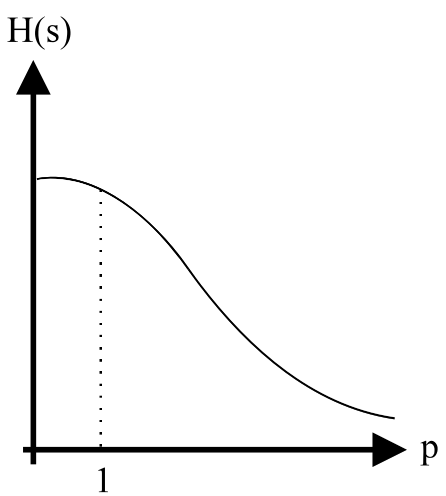
$$H(s)=\frac{1}{s+a}$$

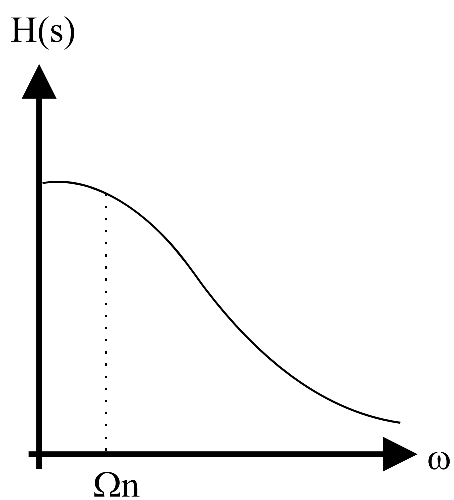
$$h(t)=e^{-at} \qquad t>0$$

- **System is stable**
- **BOUNDED OUTPUT FOR BOUNDED INPUT**

Properties of Network Functions:

Frequency Transformation





For inductors and capacitors:

$$jp(L) \Rightarrow j\omega\left(\frac{L}{\Omega_n}\right)$$

$$jp(C) \Rightarrow j\omega\left(\frac{C}{\Omega_n}\right)$$

- Most of the filter approximations are normalized to 1 rad/sec. Hence, it is necessary to denormalize the transfer function.
- **Using the frequency transformation**

$$\omega = \frac{p}{\Omega_n}$$

-
- 1 rad/sec is translated to
- $1/\Omega_n$ rad/sec

Impedance denormalization

- Typically the network elements are normalized to 1 Ω. Hence an impedance denormalization scheme must be used

$$Z \Rightarrow \Omega_n Z$$

- or

$$R \Rightarrow \Omega_n R$$

$$L \Rightarrow \Omega_n L$$

$$C \Rightarrow C/\Omega_n$$

- Note that the transfer function is invariant with the impedance denormalization (RC and LC products remain constant!!!!)

- In general both frequency and impedance denormalizations are used

Filter Approximation: Maximally flat

$$N(j\omega)=K\frac{\left(1-b_2\omega^2+b_4\omega^4-\cdots\right)+jb_1\omega\left(1-b_2{}'\omega^2+b_4{}'\omega^4-\cdots\right)}{\left(1-a_2\omega^2+a_4\omega^4-\cdots\right)+ja_1\omega\left(1-a_2{}'\omega^2+a_4{}'\omega^4-\cdots\right)}$$

Maximally flat is characterized by $a_i=b_i$ for as many coefficients as possible

- An example is the Butterworth function (lowpass) in which most of the coefficients are zero. In the following expression "n" is the filter order and ε is the passband ripple

$$N(j\omega)=\frac{1}{1+\varepsilon s^n}$$

- Lowpass filter
- Unity dc gain
  gain @ 1 Hz is determined by ε

$$\left|N(j\omega)\right|^2=N(j\omega)\times N(j\omega)^*=\frac{1}{1+\varepsilon^2\omega^{2n}}$$

Poles of the system: Maximally flat (e=1)

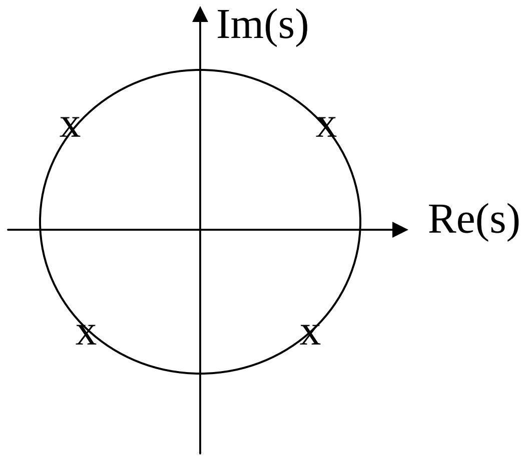
$$|N(j\omega)|^2 = \frac{1}{1+\omega^{2n}} = \frac{1}{1+(-1)^n s^{2n}}$$

Poles at: $s_p = \left[-(-1)^{-n}\right]^{1/2n} = e^{jk\pi\left(\frac{1-n}{2n}\right)}$ $\quad k=\pm1,\pm3,..$

➪Poles are located on the unit circle.
The real and imaginary parts are:

$$\mathrm{Re}(s_p) = -\sin\frac{k}{2n}\pi \qquad k=\pm1,\pm3,..$$

$$\mathrm{Im}(s_p) = \cos\frac{k}{2n}\pi$$



Design Example

For a 1 KHz lowpass filter with: Ripple=0.1, attenuation at 10 kHz>1000

Normalized prototype:ωc=1 rad/sec, ωs=10 rad/sec

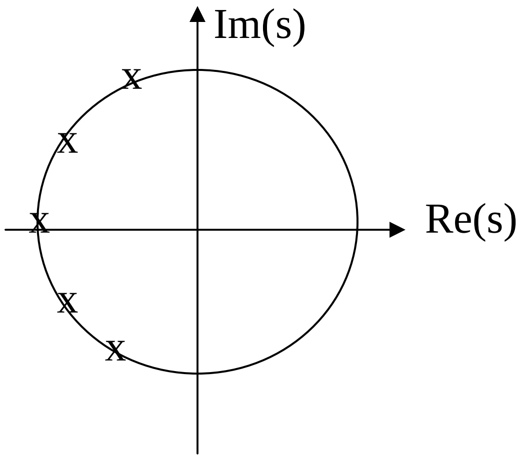
x

$$|N(j\omega)|^2 = \frac{1}{1+(0.1)^2 10^{2n}} \le \frac{1}{\left(10^3\right)^2} \qquad 10^{2n} \ge 10^8 \quad \text{or} \quad n \ge \log_{10}(10^4) = 4$$

➪**N=5**

$$\mathrm{Re}(s_p) = -\frac{1}{\varepsilon^2}\left(\sin\frac{k}{10}\pi\right) \qquad k=\pm1,\pm3,\pm5$$

$$\mathrm{Im}(s_p) = \frac{1}{\varepsilon^2}\left(\cos\frac{k}{10}\pi\right)$$



Filter Approximation: Equal Ripple

$$N(j\omega)=\frac{1}{1+\varepsilon s^{n}} \qquad \left|N(j\omega)\right|^{2}=N(j\omega)\times N(j\omega)^{*}=\frac{1}{1+\varepsilon^{2}\omega^{2n}}$$

In general

$$N(j\omega)=\frac{1}{1+\varepsilon C_{n}{}^{n}} \qquad \left|N(j\omega)\right|^{2}=N(j\omega)\times N(j\omega)^{*}=\frac{1}{1+\varepsilon^{2}C_{n}{}^{2}}$$

If Chebyshev polynomials are used ➪

$$C_{n}=\cos(n\cdot\cos^{-1}(\omega)) \quad = \quad \frac{e^{j\phi}+e^{-j\phi}}{2}$$

Filter Approximation: Equal Ripple

$$C_n = \cos(n \cdot \cos^{-1}(\omega)) \quad = \quad \frac{e^{jn\phi} + e^{-jn\phi}}{2}$$

➪**Cn > 1 if ϕ is complex. This is the case if ω>1**

since $\phi = \cos^{-1}(\omega)$

and $\cos(n\phi) = \cosh(jn\phi)$

$\omega = \cosh(j\phi), j\phi = \cosh^{-1}(\omega)$

then $C_n = \cosh(n \cdot \cosh^{-1}(\omega))$

n	C_n
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
n	$2\omega C_{n-1} - C_{n-2}$

Properties of the Chebyshev polynomials

$$C_n = \cos(n \cdot \cos^{-1}(\omega)) \le 1 \quad \omega \le 1$$

Faster response in the stopband

➪For n>3 and ω>1

In the passband, ω<1, Cn is limited to ±1, then, the ripple is determined by ε

	Butterworth	Chebyshev
$\frac{\partial}{\partial\omega} C_n^2$	$2n\omega^{2n-1}$	$\approx (2^3)^{n-1}\omega^{2n-1}$

$$|N(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2}$$

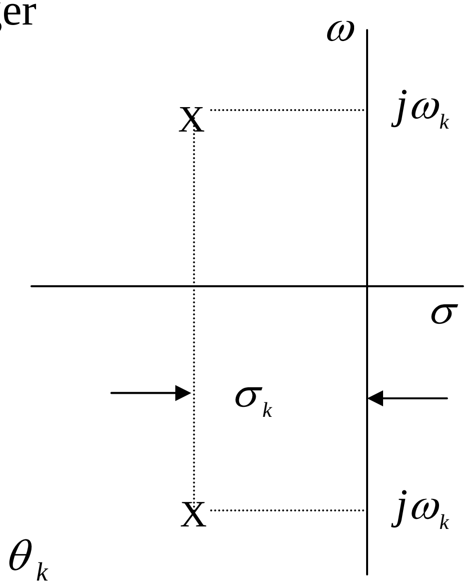
The -3 dB frequency can be found as:

$$\varepsilon^2 C_n^2(\omega_{-3dB}) = 1 \qquad \omega > 1$$

$$\omega_{-3dB} = \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right) \qquad \omega > 1$$

Comparison of Steps for Maximally Flat and Chebyshev Cases

Maximally Flat	Step	Chebyshev
$n = \dfrac{\log\left[\left(10^{\alpha_{mi}} - 1\right)/\left(10^{\alpha_{\max}/10} - 1\right)\right]}{2\log\omega_s}$ Round up to an integer	1. Find n	$n = \dfrac{\cosh^{-1}\left[\left(10^{\alpha_{\min}/10} - 1\right)/\left(10^{\alpha_{\max}/10} - 1\right)\right]^{1/2}}{\cosh^{-1}\omega_s}$ Round up to an integer
$\varepsilon = \left(10^{\alpha_{\max}/10} - 1\right)^{1/2}$	2. Find ε	$\varepsilon = \left(10^{\alpha_{\max}/10} - 1\right)^{1/2}$
a. If n is odd $\theta_k = 0^o, \quad \pm k\dfrac{180^o}{n}$, k is an integer b. If n is even, $\theta_k = \pm\dfrac{180^o}{2n}, \quad \Omega \pm \dfrac{180^o}{n} \pm \dfrac{180}{2n}$ Radius $= \Omega_0$ $-\sigma_k = \Omega_0\cos\theta_k, \quad \pm\omega_k = \Omega_0\sin\theta_k$	3. Find pole locations	a. Find θ_k for Butterworth case b. Find $\alpha = \dfrac{1}{n}\sinh^{-1}\left(\dfrac{1}{\varepsilon}\right)$ c. Then $-\sigma_k = \sin\theta_k \sinh\alpha$ $\pm\omega_k = \cos\theta_k \cosh\alpha$



The Inverse Chebyshev Approximation

$$\left|T_n(j\omega)\right|^2 = \left.\frac{1}{1+\left|K(j\omega)\right|^2}\right|_{K(j\omega)=\varepsilon C_n(\omega)} = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

$$1\left|T_c(j\omega)\right|^2 = \left.\frac{\varepsilon^2 C_n^2[\omega]}{1+\varepsilon^2 C_n^2(\omega)}\right|_{\omega\to 1/\omega} = \left|T_{I_c}(j\omega)\right|^2 = \frac{\varepsilon^2 C_n(1/\omega)}{1+\varepsilon^2 C_n^2(1/\omega)}$$

$$C_0(\omega)=1 \quad , \qquad C_3(\omega)=4\omega^3-3\omega$$

$$C_1(\omega)=\omega \qquad C_n(\omega)=2\omega C_{n-1}(\omega)-C_{n-2}(\omega)$$

$$C_2(\omega)=2\omega^2-1 \quad C_n\left(\frac{1}{\omega}\right)\cong 2^{n-1}(1/\omega)^n \;\;,\;\; \omega<<1$$

$$\alpha=-10\log\left\{1+\frac{1}{\varepsilon^2 C_n^2(1/\omega)}\right\} db$$

INVERSE CHEBYSHEV

$$\alpha_{\max} = 10 \log\left[1 + \frac{1}{\varepsilon^2 C_n^2\left(1/\omega_p\right)}\right]$$

$$C_n^2\left(\frac{1}{\omega_p}\right) = \frac{10^{\alpha_{\min/10}} - 1}{10^{\alpha_{\max/10}} - 1}$$

$$\cosh\left[n \cosh^{-1}\left(\frac{1}{\omega_p}\right)\right] = \left[\frac{10^{\alpha_{\min/10}} - 1}{10^{\alpha_{\max/10}} - 1}\right]^{1/2}$$

$$n = \frac{\cosh^{-1}\left[\left(10^{\alpha_{\min/10}} - 1\right)/\left(10^{\alpha_{\max/10}} - 1\right)\right]^{1/2}}{\cosh^{-1}\left(1/\omega_p\right)}$$

$$\alpha_{\max},\ \alpha_{\min},\ \omega_p,\ \omega_s$$

$$|T|^2 = \left.\frac{p(s)\,p(-s)}{q(s)q(-s)}\right|_{s=j\omega}$$

$$p(s)\,p(-s)\Big|_{s=j\omega} = \varepsilon^2 C_n^2\left(\frac{1}{\omega}\right)$$

$$q(s)q(-s)\Big|_{s-j\omega} = 1+\varepsilon^2 C_n^2\left(\frac{1}{\omega}\right)$$

Zeros are those values ω_κ where

$$C_n^2\left(\frac{1}{\omega_k}\right) = 0$$

$$\omega_k = \sec\left(\frac{k_\pi}{2n}\right), \quad k = 1,3,5,\cdots,n$$

$$S_R = \sigma_k + j\omega_k$$ Pole Location (Chebyshev)

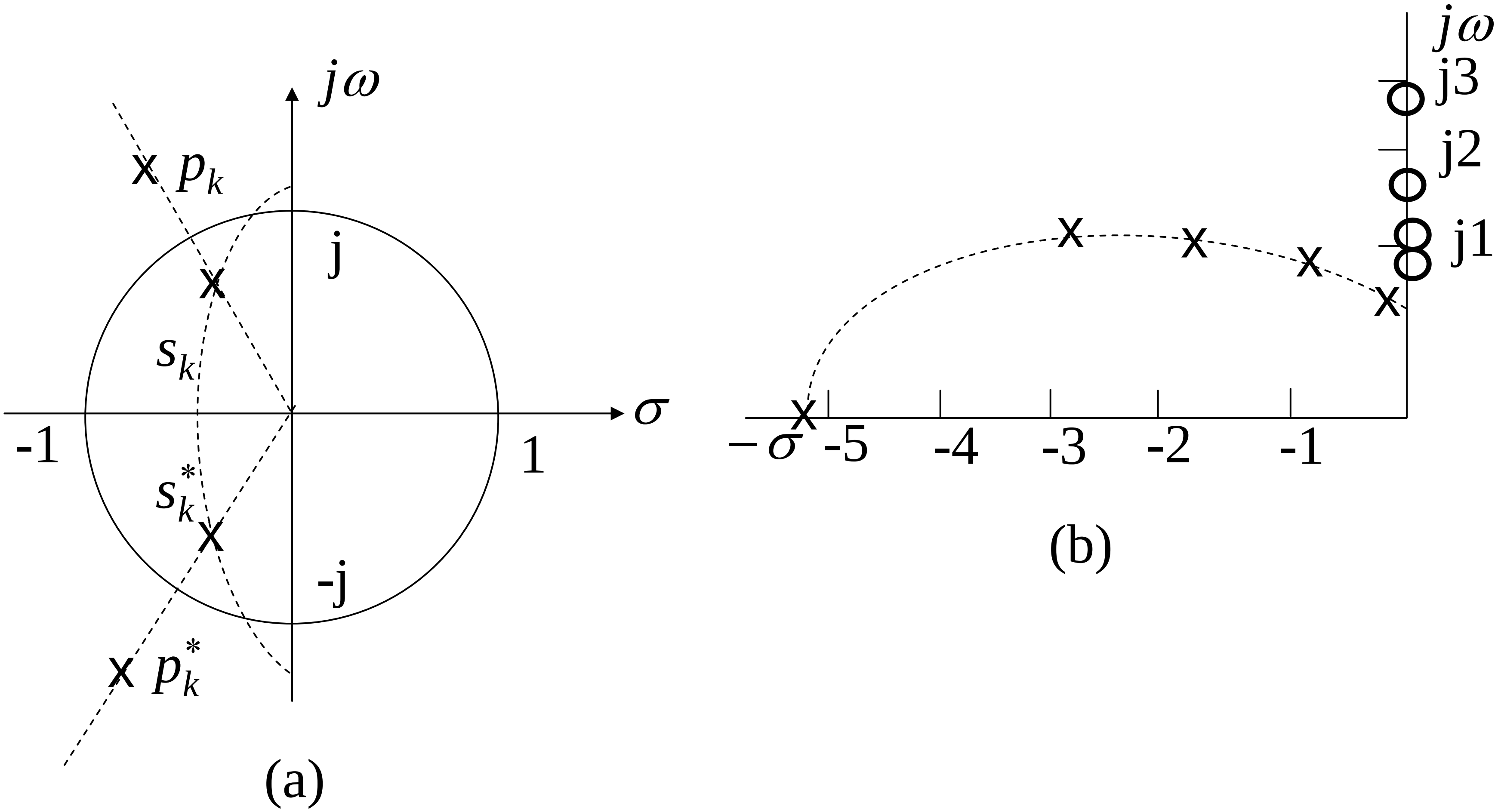
$$\omega_{0k} = \sqrt{\sigma_k^2 + \omega_k^2} \text{ and } \mathrm{Q_{Ck}} = \frac{\omega_{\mathrm{0k}}}{2\sigma_k}$$

$$p_k = \alpha_k + j\beta_k$$ Pole Location (Inverse Chebyshev)

$$p_k = \frac{1}{s_k} = \frac{\sigma_k - j\omega_k}{\sigma_k^2 + \omega_k^2} = \frac{\sigma_k}{\omega_{0k}^2} - j\frac{\omega_k}{\omega_{0k}^2} = \alpha_k + j\beta_k$$

Magnitude and quality factor

$$\left|p_k\right| = \sqrt{\alpha_k^2 + \beta_k^2} = \frac{1}{\omega_{0k}} \text{ and } Q_{ICk} = \frac{\left|\mathrm{p_k}\right|}{2_{\alpha_k}} = \frac{1/\omega_{0k}}{2\sigma_k / \omega_{0k}^2} = \frac{\omega_{0k}}{2\sigma_k} = Q_{Ck}$$



(a) Pole reciprocation in an inverse Chebyshev filter; since the poles of the Chebyshev and the inverse Chebyshev filter lie on the same radial line, they have the same pole Q. (b) Location of poles and zeros of a ninth-order inverse Chebyshev filter.

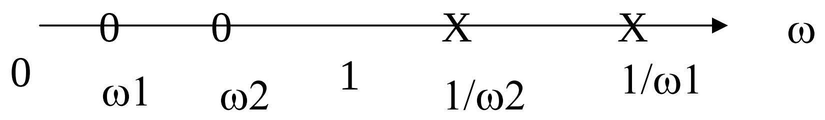
Properties of Elliptic Filters

$$\left|N(j\omega)\right|^2 = \frac{1}{1+R_n^2(\omega)}$$

$$R_n(\omega) = \frac{\left(\omega_1^2-\omega^2\right)\left(\omega_2^2-\omega^2\right)\ldots\left(\omega_{n/2}^2-\omega^2\right)}{\left(1-\omega_1^2\omega^2\right)\left(1-\omega_2^2\omega^2\right)\ldots\left(1-\omega_{n/2}^2\omega^2\right)}, \qquad n \text{ even}$$

$$R_n(\omega) = \frac{\omega\left(\omega_1^2-\omega^2\right)\left(\omega_2^2-\omega^2\right)\ldots\left(\omega_{(n-1)/2}^2-\omega^2\right)}{\left(1-\omega_1^2\omega^2\right)\left(1-\omega_2^2\omega^2\right)\ldots\left(1-\omega_{(n-1)/2}^2\omega^2\right)}, \qquad n \text{ odd}$$

Note that $Rn(1/\omega)=1/Rn(\omega)$



Properties of Elliptic Filters

$$|N(j\omega)|^2 = \frac{1}{1 + R_n^{\,2}(\omega)}$$

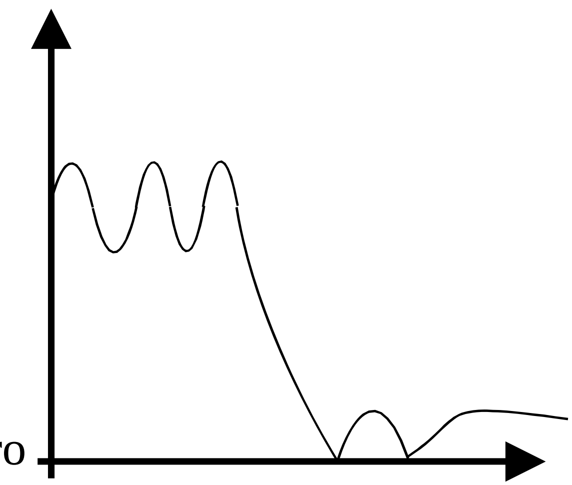
In the most general case (ω_s is the frequency at which the stop band begins)

$$R_n(\omega) = M\frac{\left(\omega_1^{\,2} - \frac{\omega^2}{\omega_s}\right)\left(\omega_2^{\,2} - \frac{\omega^2}{\omega_s}\right)\cdots\left(\omega_{n/2}^{\,2} - \frac{\omega^2}{\omega_s}\right)}{\left(1 - \frac{\omega_1^{\,2}\omega^2}{\omega_s}\right)\left(1 - \frac{\omega_2^{\,2}\omega^2}{\omega_s}\right)\cdots\left(1 - \frac{\omega_{n/2}^{\,2}\omega^2}{\omega_s}\right)}, \qquad n \ \text{ even}$$

$$R_n(\omega) = M\frac{\omega\left(\omega_1^{\,2} - \frac{\omega^2}{\omega_s}\right)\left(\omega_2^{\,2} - \frac{\omega^2}{\omega_s}\right)\cdots\left(\omega_{(n-1)/2}^{\,2} - \frac{\omega^2}{\omega_s}\right)}{\left(1 - \frac{\omega_1^{\,2}\omega^2}{\omega_s}\right)\left(1 - \frac{\omega_2^{\,2}\omega^2}{\omega_s}\right)\cdots\left(1 - \frac{\omega_{(n-1)/2}^{\,2}\omega^2}{\omega_s}\right)}, \qquad n \ \text{ odd}$$

Properties of Elliptic Filters

$$N(s) = \sqrt{\frac{1}{1+R_n^{\,2}(\omega)}} = \frac{H_0 \prod \left(s^2 - \Omega_i^{\,2}\right)}{D\left(s,\ldots s^n\right)}$$



- n/2 peaks in the passband
- n/2 zeros in the stop band

- In the case of odd-order filters,
- (n-1)/2 zeros in the stop band

- Note that Rn(∞) is finite, then N (∞) is not zero
- (case A).

- In other filters, the number of zeros is reduced,
- then N (∞)=0 (case B).

Filter Approximation: Properties of Linear Phase Filters

• Important in hard-disk applications and phase equalizers

•for typical applications, 7th-order LP filters are required

•For these filters, the most important parameter is the group delay and PHASE

•In the passband the magnitude can be scaled and the signals can be delayed but by the same amount of time

$$v_{out}(t) = Kv_{in}(t - t_0)$$

$$V_{out}(s) = V_{in}(s)\left[Ke^{-j\omega(t_0)}\right]$$

Filter Approximation: Properties of Linear Phase Filters

$$H(s) = \left[Ke^{-j\omega(t_0)}\right] \qquad \phi(s) = \omega t_0 (\text{linear} \quad \text{phase})$$

For a typical lowpass filter

$$N(s) = K\frac{1}{1 + a_1 s + a_2 s^2 + ..} = \frac{K}{1 - a_2\omega^2 + .. + j\omega\left(a_1 - a_4\omega^2 + ..\right)},$$

and

$$\phi(\omega) = \arg(N(j\omega)) = \tan^{-1}\left(-\frac{\omega\left(a_1 - a_3\omega^2 + ..\right)}{1 - a_2\omega^2 + ..}\right)$$

Filter Approximation: Properties of Linear Phase Filters

o using the following series expansion $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$$\arg(N(j\omega)) = \tan^{-1}\left(-\frac{\omega\left(a_1 - a_3\omega^2 + ..\right)}{1 - a_2\omega^2 + ..} \right)$$

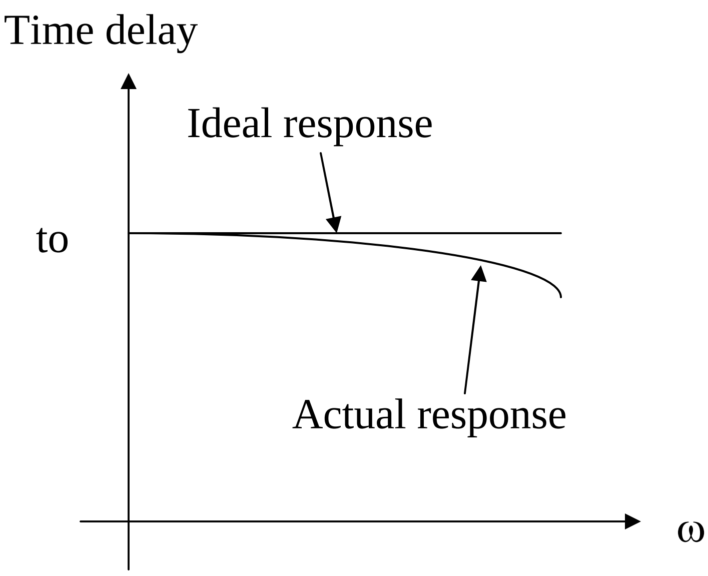
The condition for linear phase is:

$$\frac{\partial}{\partial\omega}\tan^{-1}(x) = \frac{\partial}{\partial\omega}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \right) = \text{cons}\tan\text{t}$$

These are called Bessel polynomials, and the resulting networks are called the Thomson filters.

Filter Approximation: Properties of Linear Phase Filters

➪ Typically the phase behavior of the filters present some some deviations



Errors are measured in time

The important parameter is the GROUP DELAY FUNCTION

$$\text{groupdelay} = \frac{\partial}{\partial \omega}(\phi)$$

DELAY EQUALIZATION

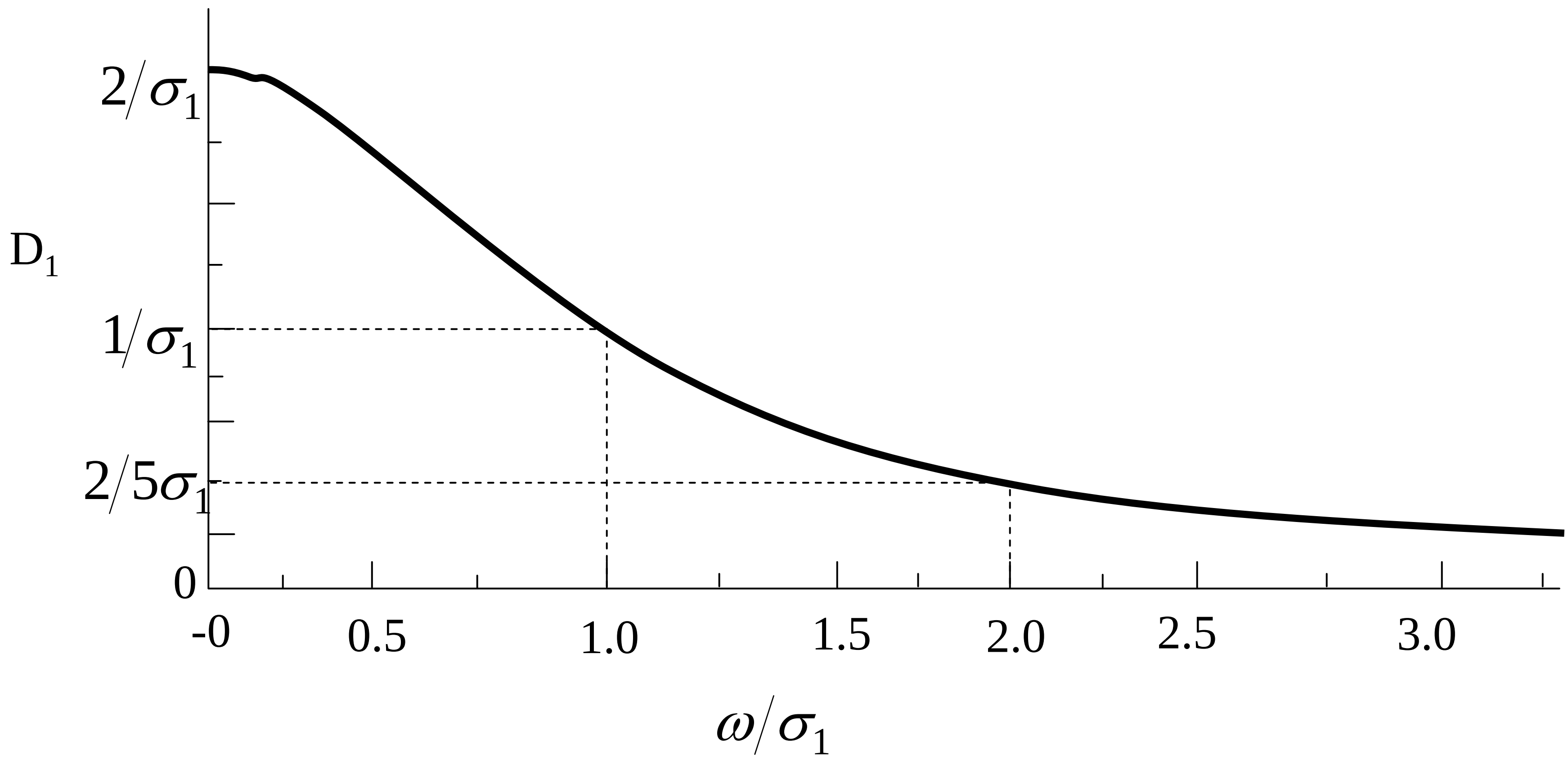
First-Order All-pass Function

$$T(s) = K\frac{s-\sigma_1}{s+\sigma_1}$$

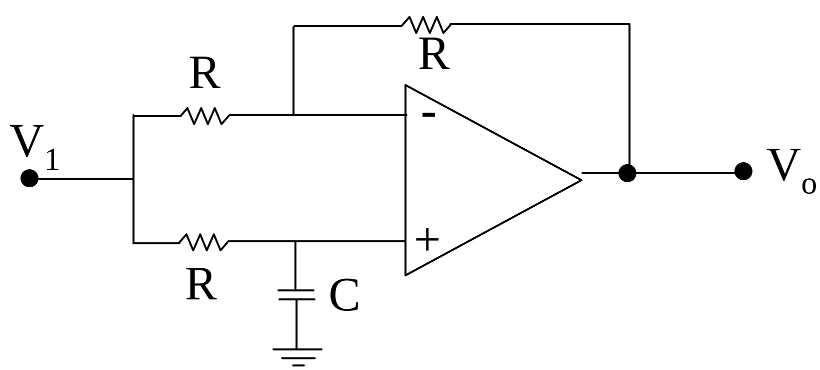
$$|T_1(j\omega)| = K \quad \text{and} \quad \theta_1 = \theta_{numerator} - \theta_{denominator} = -2\tan^{-1}\left(\frac{\omega}{\sigma_1}\right)$$

$$D_1(\omega) = -\frac{d\theta}{d\omega} = \frac{2/\sigma_1}{1+(\omega/\sigma_1)^2}$$

$$D_{1,\max} = D_1(0) = 2/\sigma_1$$



Delay of a first-order filter.



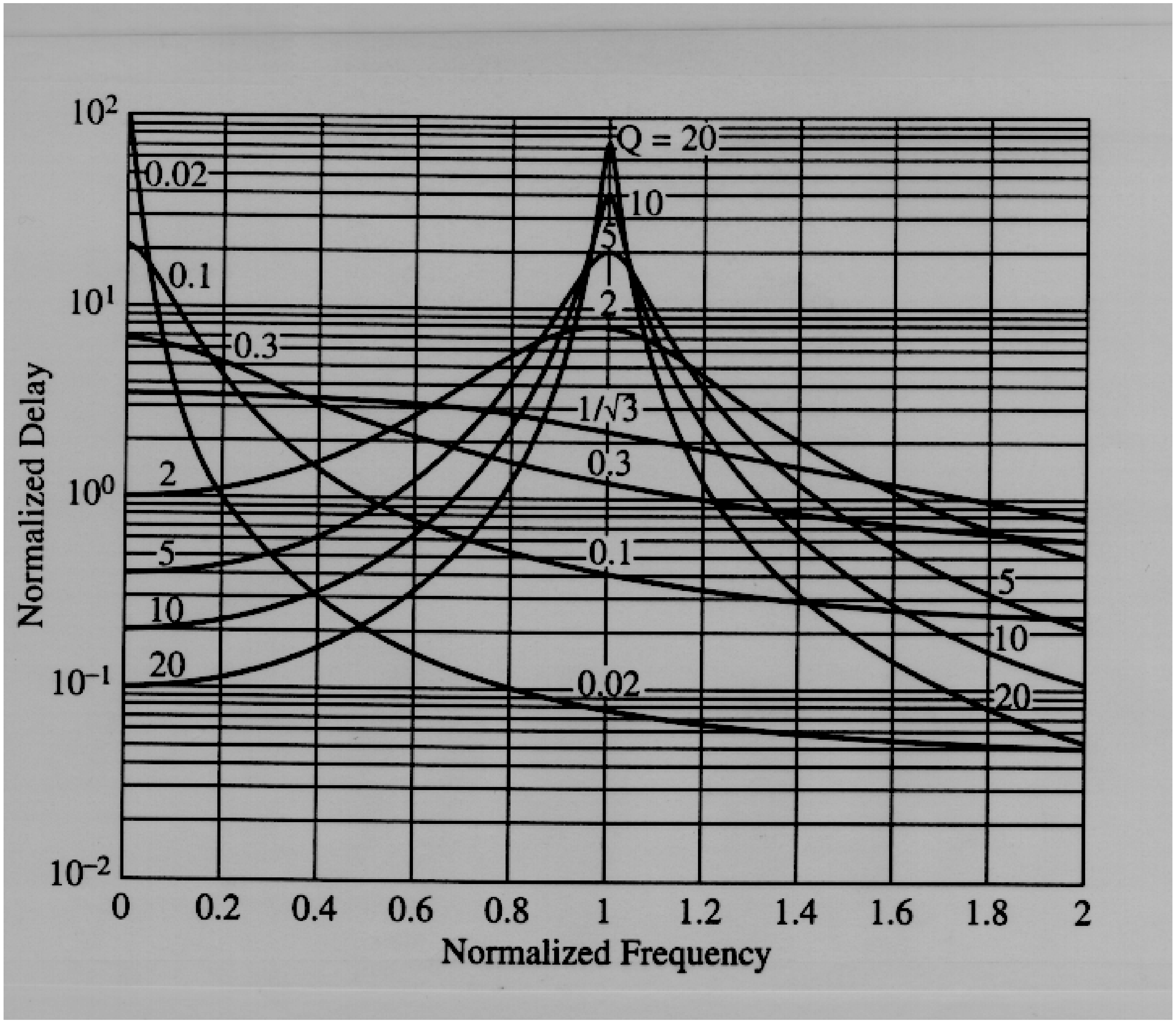
SECOND-ORDER EQUALIZATION

$$T_1(s) = K_0 \frac{s^2 - s\omega_0 / Q + \omega_0^2}{s^2 + s\omega_0 / Q + \omega_0^2} \quad , \quad T_2(s_n) = K_0 \frac{s_n^2 - s_n / Q + 1}{s_n^2 + s_n / Q + 1}$$

$$\theta_2(\omega_n) = -2\tan^{-1}\left(\frac{\omega_n / Q}{1-\omega_n^2}\right)$$

$$D_2(\omega_n) = -\frac{d\theta_2(\omega_n)}{d\omega} = -\frac{d\theta_2(\omega_n)}{d\omega_n} \times \frac{d\omega_n}{d\omega} = \frac{d}{d\omega_n}\left[2\tan^{-1}\left(\frac{\omega_n / Q}{1-\omega_n^2}\right)\right] \times \frac{1}{\omega_0}$$

$$D_2(\omega_n) = \frac{1}{\omega_0}\left[\frac{(2/Q)\left(1+\omega_n^2\right)}{\left(1-\omega_n^2\right)^2 + (\omega_n / Q)^2}\right]$$



The normalized delay $\omega_0 D_2(\omega_\nu)$ of a second-order allpass filter as function of Q

$$D_2(0) = \frac{1}{\omega_0}\frac{2}{Q}$$

$$\left(\omega_n^2\right)^2 + 2\omega_n^2 + \left(Q^{-2} - 3\right) = 0$$

$$\omega_{n,\max} = \sqrt{-1 + \sqrt{4 - Q^{-2}}}$$

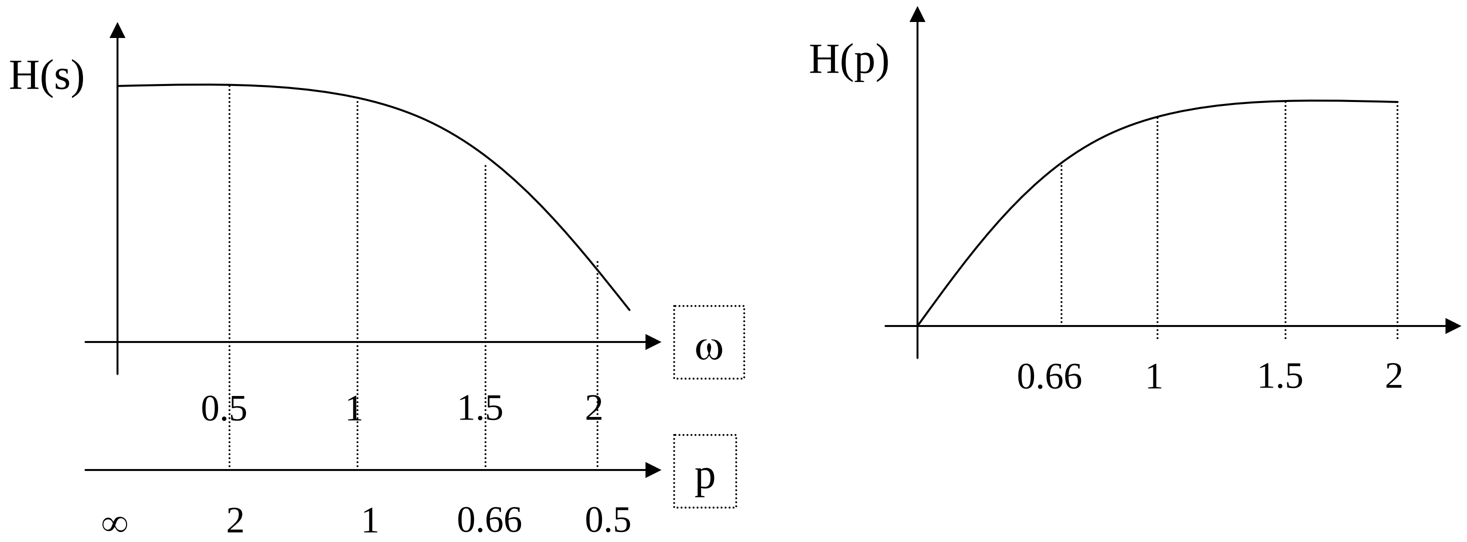
$$D_{2,\max} \approx \frac{4Q}{\omega_0} \quad , \quad D_{2,\max} = \frac{2}{\omega_0 / (2Q)} = \frac{2}{\alpha}$$

$$D_2(\omega_n) = \frac{2\sqrt{3}}{\omega_0}\left(\frac{1+\omega_n^2}{1+\omega_n^2+\omega_n^4}\right)$$

Frequency Transformations

- Lowpass to Highpass

$$s \Rightarrow \frac{1}{p} \quad \text{then} \quad H_{lp}(s) = \frac{H_0}{\sum_{i=0}^{n} a_i s^i} \quad \Rightarrow \quad H_{hp}(p) = \frac{H_0}{\sum_{i=0}^{n} a_i \left(\frac{1}{p}\right)^i} = \frac{H_0 p^n}{\sum_{i=0}^{n} a_i p^{n-i}}$$



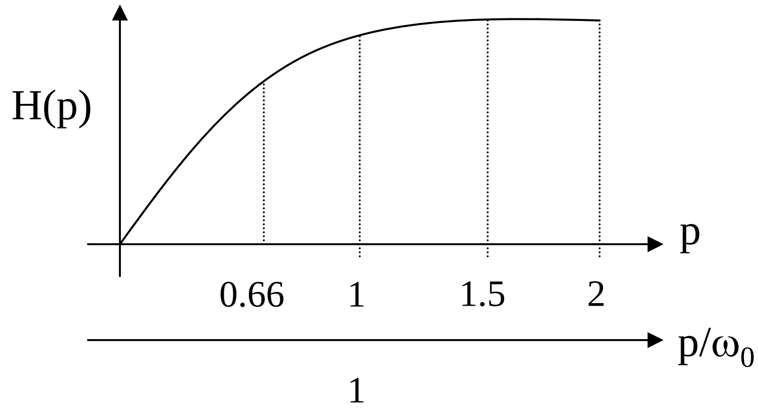
✓ Lowpass to Highpass

$$H_{lp}(s) = \frac{H_0}{\sum_{i=0}^{n} a_i s^i} \Leftrightarrow H_{hp}(p) = \frac{H_0 p^n}{\sum_{i=0}^{n} a_i p^{n-i}}$$

✓N zeros at ∞ are translated to zero
✓Poles are not the same!!!
✓The main characteristics of the lowpass filter are maintained
✓for a highpass filter with cutoff frequency at $\omega_{0,}$ then

$$s \Rightarrow \frac{\omega_0}{p}$$

This transformation scheme translates ω=1 to $p=\omega_0$



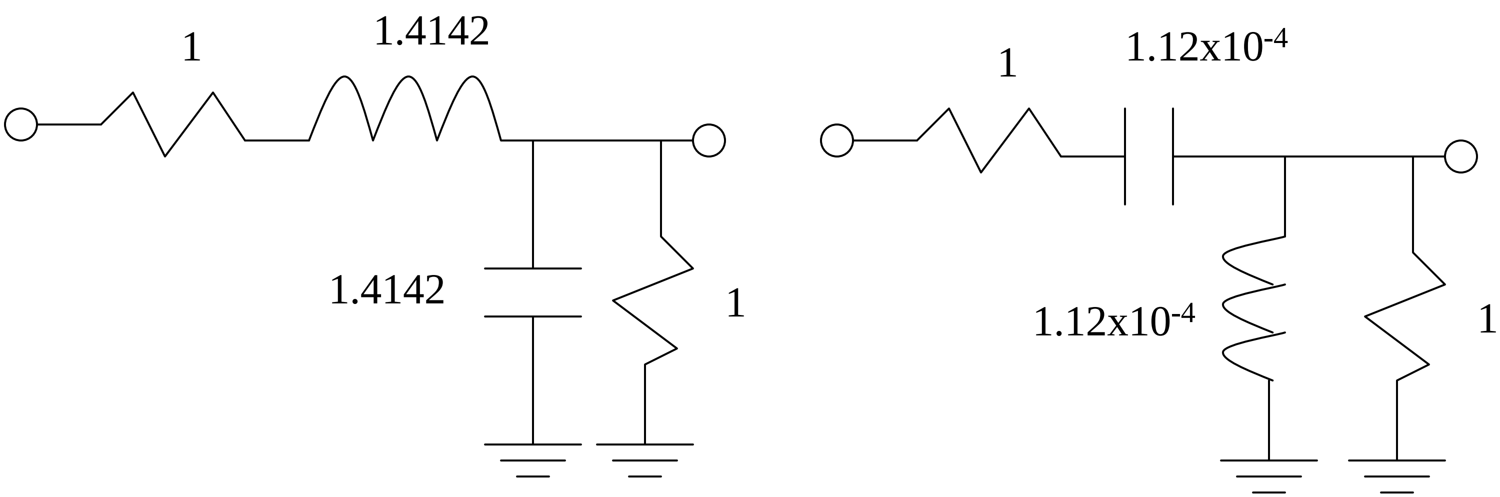
Frequency Transformations

- The Lowpass to Highpass transformation can also be applied to the elements

Element	impedance	transformed
Resistor	R	R
inductor	sL	$\frac{\omega_0 L}{p}$
capacitor	$\frac{1}{sC}$	$\frac{p}{\omega_0 C}$

- The resistors are not affected
- L is transformed in a capacitor Ceq=1/ω_0L
- C is transformed in an inductor Leq=1/ω_0C

Example: Design a 1KHz HP-filter from a LP prototype.



Frequency Transformations

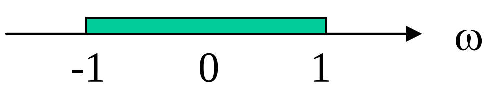
➢ Lowpass to Bandpass transformation

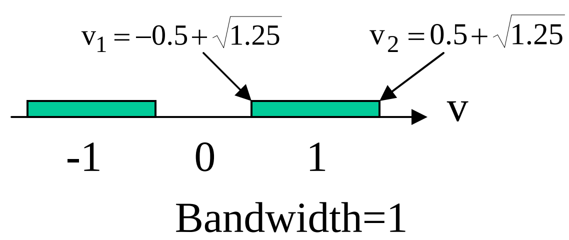
$$s \Rightarrow \frac{p^2+1}{p} \qquad \text{then} \quad H_{lp}(s) = \frac{H_0}{\sum_{i=0}^{n} a_i s^i} \quad \Rightarrow \quad H_{bp}(p) = \frac{H_0 p^n}{\sum_{i=0}^{2n} b_i (p)^i}$$

- n zeros at $\omega=0$ and n zeros at ∞
- even number of poles
- The bandwidth of the BP is equal to the bandwidt of the LP

- In the p-domain

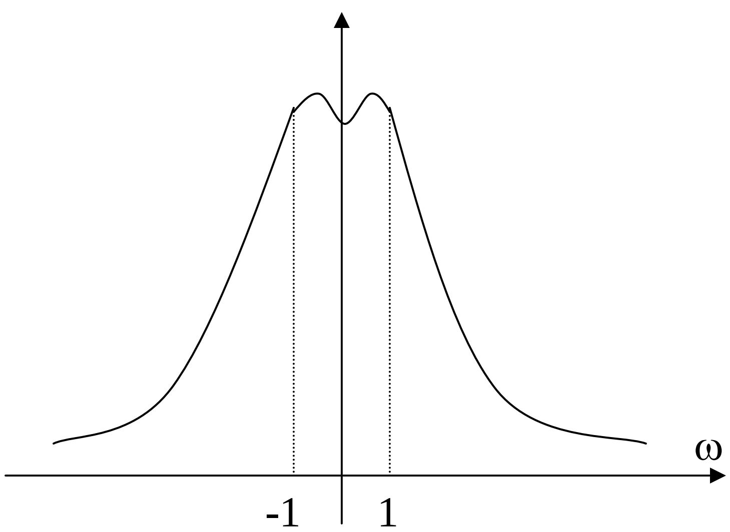
$$p = jv = \frac{s}{2} \pm \sqrt{\left(\frac{s}{2}\right)^2 - 1} \quad \text{or} \quad v = \frac{\omega}{2} \pm \sqrt{\left(\frac{\omega}{2}\right)^2 + 1}$$



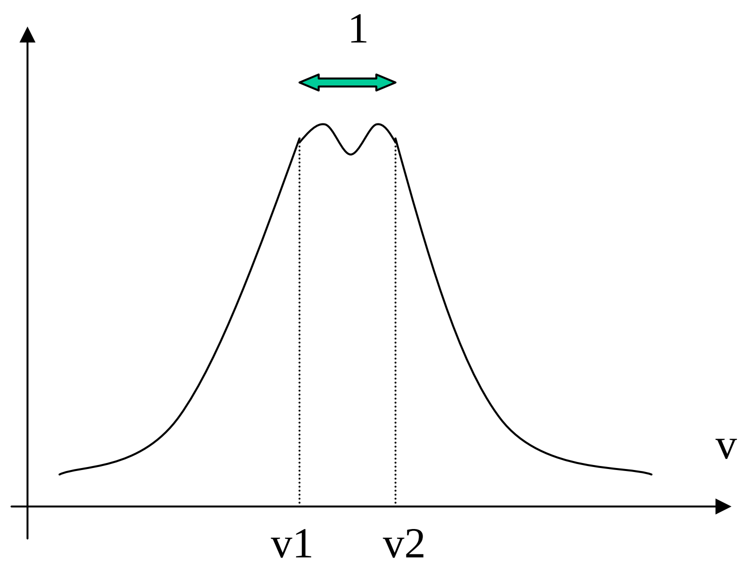


Frequency Transformations

➪Note that $v_2 - v_1 = 1$ and $v_2 \cdot v_1 = 1$



Lowpass prototype



Bandpass filter

Frequency Transformations

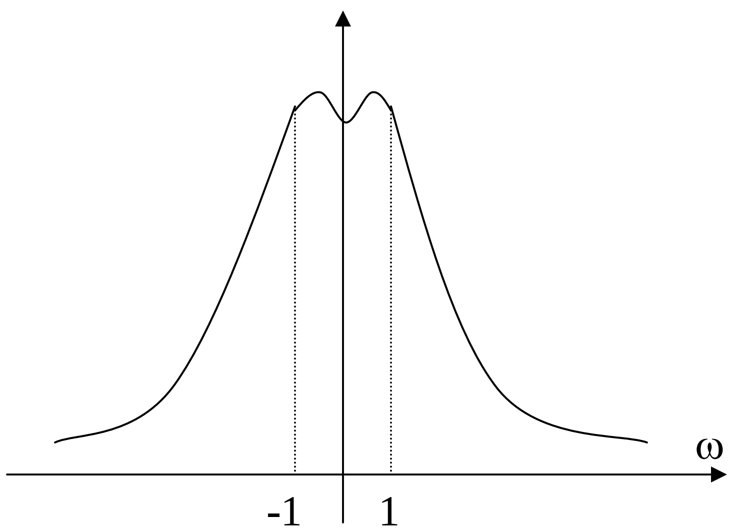
➪General transformation $s = \frac{1}{BW}\left(\frac{p^2+\omega_0{}^2}{p}\right)$

$$v = \frac{BW\cdot\omega}{2} \pm \sqrt{\left(\frac{BW\cdot\omega}{2}\right)^2 + \omega_0{}^2}$$

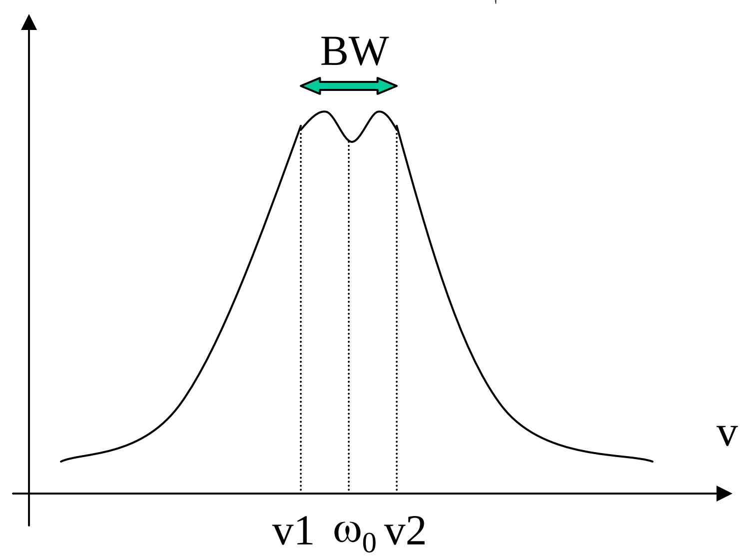
$$-1 \;\Rightarrow\; v_1 = -\frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_0{}^2}$$

$$0 \;\Rightarrow\; \omega_0$$

$$1 \;\Rightarrow\; v_2 = +\frac{BW}{2} + \sqrt{\left(\frac{BW}{2}\right)^2 + \omega_0{}^2}$$



Lowpass prototype

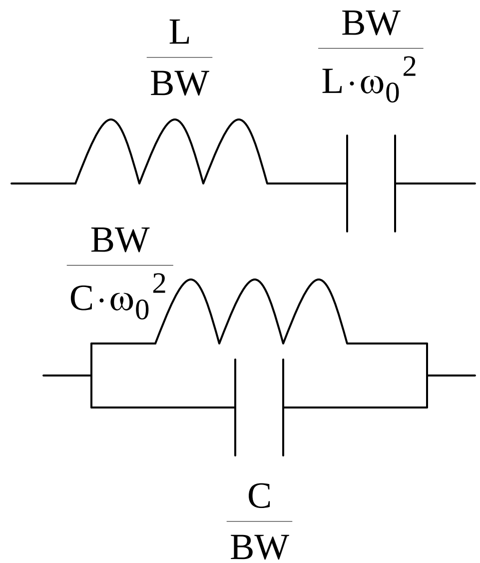


Bandpass filter

Frequency Transformations

➪The Lowpass to Bandpass transformation can also be applied to the elements

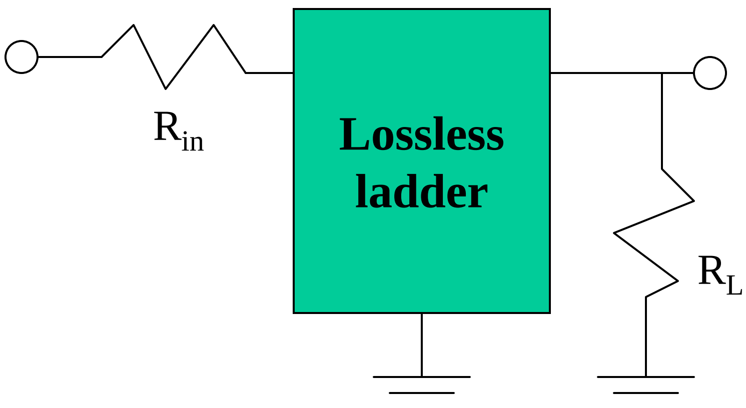
Element	impedance	transformed
Resistor	R	R
inductor	sL	$\frac{L}{BW}p+\frac{\omega_0^2 L}{BW}\frac{1}{p}$
capacitor	$\frac{1}{sC}$	$\frac{1}{\frac{C}{BW}p+\frac{\omega_0^2 C}{BW}\frac{1}{p}}$



- **Note that for $\omega=\omega_0$**
- **for the inductor Zeq=0**
- **for the capacitor Yeq=0 (Zeq=∞)**

Frequency Transformations

- **In general, for double-resistance terminated ladder filters**



➢ **around $\omega=\omega_0$**

$$H(s)\Big|_{\omega=\omega_0} = \frac{R_L}{R_{in}+R_L} = \frac{1}{1+\dfrac{R_L}{R_{in}}}$$

➢ **In the passband, the transfer function can be very well controlled**

➢**Low-sensitivy**

Frequency Transformations

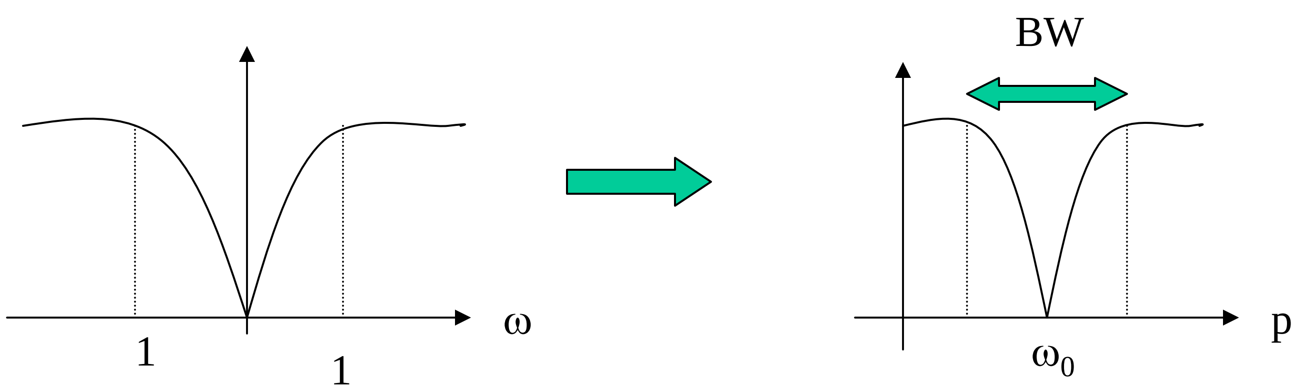
- Lowpass to Bandreject transformation

$$s \Rightarrow \frac{1}{\frac{1}{BW}\left(\frac{p^2+\omega_0^{\,2}}{p}\right)}$$

- Lowpass to Highpass transformation (notch at $\omega=0$)

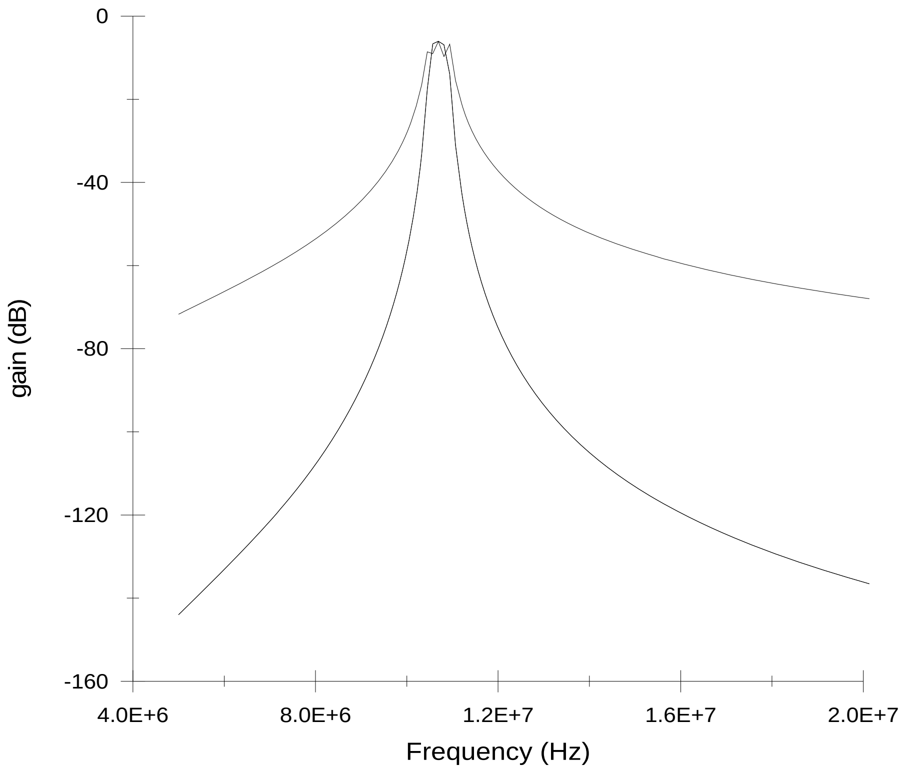
- Shifting the frequency to ω_0 and adjusting the bandwidth to BW

. Bandpass transformation!!!



Ladder Filters

- The ladder filter realization can be found in tables and/or can be obtained from FIESTA
- The elements must be transformed according to the frequency and impedance normalizations



Sensitivity

- Definition $S_x^y = \frac{x}{y}\frac{\partial y}{\partial x}$ Y= transfer function and x = variable or element

Some properties:

$$S_x^{ky} = S_{kx}^y = S_x^y \qquad S_{kx}^x = S_x^x = 1$$

$$S_x^{1/y} = S_{1/x}^y = -S_x^y \qquad S_x^{y^n} = nS_x^y$$

$$S_{x^n}^y = \frac{1}{n}S_x^y \qquad S_x^y = S_{x_2}^y S_x^{x_2}$$

$$S_x^{\prod_{i=1}^n y_i} = \sum_{i=1}^n S_x^{y_i} \qquad S_x^{\sum_{i=1}^n y_i} = \frac{\sum_{i=1}^n y_i S_x^{y_i}}{\sum_{i=1}^n y_i}$$

For a typical H(s)

$$S_{a_j}^{H(s)=\frac{H_0}{\sum a_i (j\omega)^i}} = -\frac{\sum a_i (j\omega)^i S_{a_j}^{a_i (j\omega)^i}}{\sum a_i (j\omega)^i} = -\frac{a_j (j\omega)^j}{\sum a_i (j\omega)^i}$$

Sensitivity

• Sensitivity is a **measure of the change in the performance of the system** due to a **change in the nominal value of a certain element.**

$$S_x^y = \frac{x}{y}\frac{\partial y}{\partial x} \quad \Rightarrow \quad S_x^y \cong \frac{x}{y}\frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{\Delta y}{y} = \left[S_x^y\right]\frac{\Delta x}{x}$$

⇒ Normalized variations at the output are determined by the sensitivity function and the normalized variations of the parameter

Example:

If the senstivity function is 10, then variations of Δx/x=0.01(1%) produce Δy/y=0.1(10%)

For a good design, the sensitivity functions should be < 5.
Effects of the partial positive feedback (negative resistors)?

Sensitivity

- For a typical amplifier $A_v = \frac{g_m}{g_0}$

$$S_{g_m}^{A_v} = 1, \quad S_{g_0}^{A_v} = -1$$

•Sometimes the dc gain is enhanced by using a negative resistor

$$A_v = \frac{g_m}{g_0 - g_{02}} = \frac{g_m}{g_0} \frac{1}{1 - \frac{g_{02}}{g_0}}$$

For large dc gain $g_{02}=g_0$

$$S_{g_m}^{A_v} = 1, \quad S_{g_0}^{A_v} = -\frac{g_0 S_{g_0}^{g_0}}{g_0 - g_{02}} = -\frac{1}{1 - \frac{g_{02}}{g_0}}$$

The larger the gain improvment the larger the sensitivity!!!!

Conclusions

• There is a number of conventional filter magnitude approximations

• The choice of a particular approximation is application dependent

• Besides the magnitude specifications, there exists also a phase (group delay) specification. For this the Thompson (Bessel) approximation is used

• There are a host of Filter approximation software programs, including Matlab, Filsyn, and Fiesta2 developed at TAMU

Acknowledgment: Thanks to my colleague Dr. Silva-Martínez for providing some of the material for this presentation